Math 222

Sections 10.2, 10.4
Textbook Boyce and DiPrima.
Lecture by: José L. Pabón

We will be courteous, civil to each other.
No such thing as an ‘obvious question.
Feel free to ask / clear up any doubt to clear up

Annoucements

› Dr. Frederick: Authority on examination topics

Annoucements

- Dr. Frederick: Authority on examination topics
- Free pizza!

Annoucements

- Dr. Frederick: Authority on examination topics
- Free pizza!

Christina F Wed, Dec 6, 3:44 PM (22 hours ago)
to me

Thanks a lot. Can you also mention to the students that they will have a final review on Thursday, 12/14 5:00 PM - 8:00 PM in CKB 120 and I'll provide pizza

Best
Christina

NJIT New Jersey Institute of Technology

Christina A Frederick
Associate Professor
Mathematical Sciences
christina.a.frederick@njit.edu • (973) 596-5328

Review

CHAPTER **10**

Partial Differential Equations and Fourier Series

ential
nd
es

DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x.

Constant: $\frac{d}{dx}(c) = 0$

Sum: $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

Difference: $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$

Constant Multiple: $\frac{d}{dx}(cu) = c\frac{du}{dx}$

Product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Power: $\frac{d}{dx}x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x$ $\quad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Exponential and Logarithmic Functions

$\frac{d}{dx}e^x = e^x$ $\quad$ $\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}a^x = a^x \ln a$ $\quad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ $\quad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ $\quad$ $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2}$ $\quad$ $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$

Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\quad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\quad$ $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$ $\quad$ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$ $\quad$ $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$ $\quad$ $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}$ $\quad$ $\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}}$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Review

Partial Differential Equations and Fourier Series

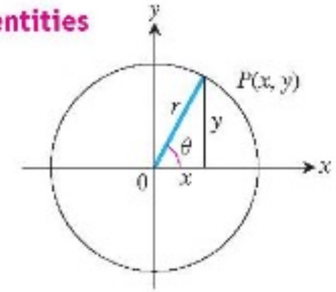
Trigonometry Formulas

1. Definitions and Fundamental Identities

Sine: $\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$

Cosine: $\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$

Tangent: $\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$



2. Identities

$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \frac{\pi}{2}\right) = \sin A$

$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$

$\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

$\sin A\cos B = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

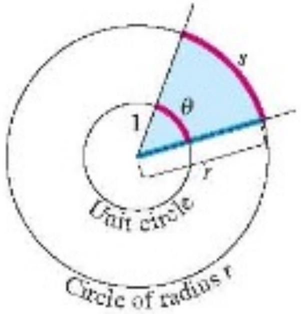
$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$

π

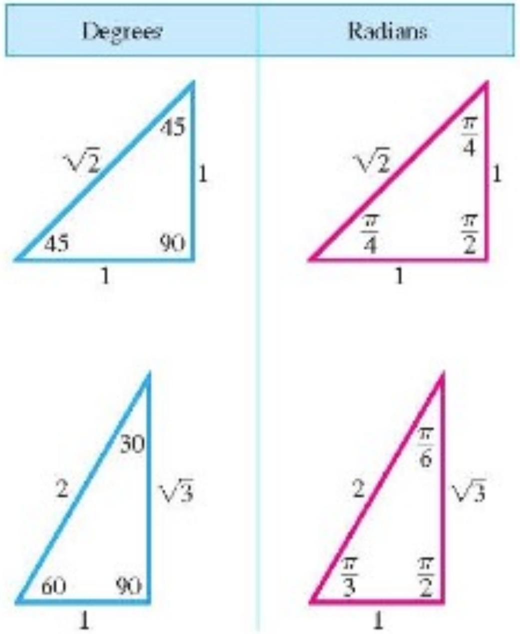
Trigonometric Functions

Radian Measure

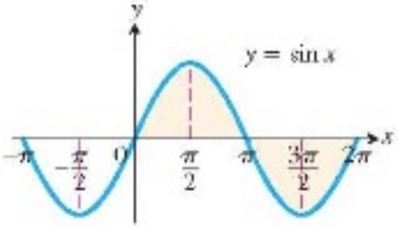


$$\frac{s}{r} = \frac{\theta}{1} = \theta \quad \text{or} \quad \theta = \frac{s}{r},$$

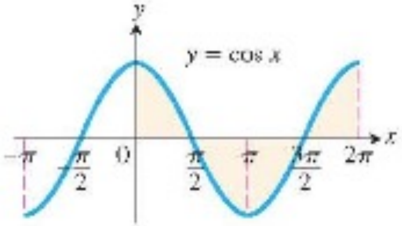
$180° = \pi$ radians.



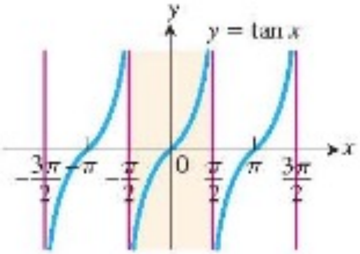
The angles of two common triangles, in degrees and radians.



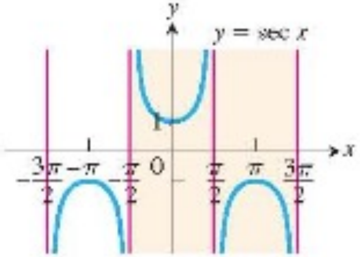
Domain: (−∞, ∞)
Range: [−1, 1]



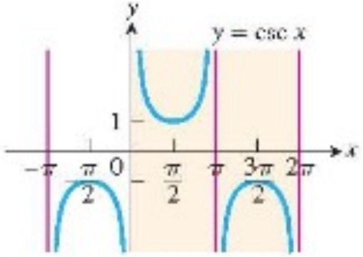
Domain: (−∞, ∞)
Range: [−1, 1]



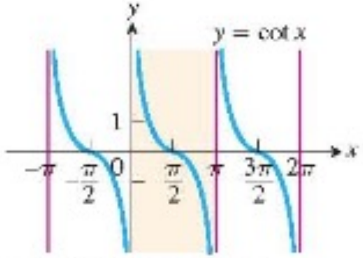
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: (−∞, ∞)



Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: (−∞, ∞)

Review

Values of Selected Trigonometric Functions

Angle in Degrees	*Sine (sin)*	*Cosine (cos)*	*Tangent (tan)*
0	0	1	0
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	1	0	Undefined
180	0	-1	0
270	-1	0	Undefined

$$\sinh(0) = 0$$

$$\cosh(0) = 1$$

$$\tanh(0) = 0$$

$$\coth(0) = \infty$$

$$\operatorname{csch}(0) = \infty$$

$$\operatorname{sech}(0) = 1$$

Review

www.konkur.in

CHAPTER **10**

Partial Differential Equations and Fourier Series

10.1 – Review

CHAPTER 10

Partial Differential Equations and Fourier Series

10.1 – Review

Physical applications often lead to another type of problem, one in which the value of the dependent variable y or its derivative is specified at two *different* points. Such conditions are called **boundary conditions** to distinguish them from initial conditions that specify the value of y and y' at the *same* point. A differential equation and suitable boundary conditions form a **two-point boundary value problem**. A typical example is the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{3}$$

with the boundary conditions

$$y(\alpha) = y_0, \quad y(\beta) = y_1. \tag{4}$$

10.1 – Review

Physical applications often lead to another type of problem, one in which the value of the dependent variable y or its derivative is specified at two *different* points. Such conditions are called **boundary conditions** to distinguish them from initial conditions that specify the value of y and y' at the *same* point. A differential equation and suitable boundary conditions form a **two-point boundary value problem**. A typical example is the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{3}$$

with the boundary conditions

$$y(\alpha) = y_0, \quad y(\beta) = y_1. \tag{4}$$

Eigenvalue Problems. Recall the matrix equation

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{17}$$

that we discussed in Section 7.3. Equation (17) has the solution $\mathbf{x} = \mathbf{0}$ for every value of λ, but for certain values of λ, called eigenvalues, there are also nonzero solutions, called eigenvectors. The situation is similar for boundary value problems.

Consider the problem consisting of the differential equation

$$y'' + \lambda y = 0, \tag{18}$$

$$y'' + \mu^2 y = 0.$$

$$y'' + \mu^2 y = 0.$$

Case I: $\lambda > 0$. To avoid the frequent appearance of radical signs, it is convenient in this case to let $\lambda = \mu^2$ and to rewrite equation (18) as

$$y'' + \mu^2 y = 0. \tag{20}$$

The characteristic polynomial equation for equation (20) is $r^2 + \mu^2 = 0$ with roots $r = \pm i\mu$, so the general solution is

$$y = c_1 \cos(\mu x) + c_2 \sin(\mu x). \tag{21}$$

Note that μ is nonzero (since $\lambda > 0$) and there is no loss of generality if we also assume that μ is positive. The first boundary condition requires that $c_1 = 0$, and then the second boundary condition reduces to

$$c_2 \sin(\mu\pi) = 0. \tag{22}$$

10.1 – Review

$$y'' + \mu^2 y = 0.$$

Case II: $\lambda < 0$. In this case we let $\lambda = -\mu^2$, so that equation (18) becomes

$$y'' - \mu^2 y = 0. \tag{25}$$

The characteristic equation for equation (25) is $r^2 - \mu^2 = 0$ with roots $r = \pm\mu$, so its general solution can be written as

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x). \tag{26}$$

$$y'' + \mu^2 y = 0.$$

Case III: $\lambda = 0$. Now equation (18) becomes

$$y'' = 0, \tag{27}$$

and its general solution is

$$y = c_1 x + c_2. \tag{28}$$

In later sections of this chapter, we will often encounter the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \tag{29}$$

which differs from the problem (18), (19) only in that the second boundary condition is imposed at an arbitrary point $x = L$ rather than at $x = \pi$. The solution process for $\lambda > 0$ is exactly the same as before, up to the step where the second boundary condition is applied. For problem (29) this condition requires that

$$c_2 \sin(\mu L) = 0 \tag{30}$$

rather than equation (22), as in the former case. Consequently, μL must be an integer multiple of π, so $\mu = n\pi/L$, where n is a positive integer. Hence the eigenvalues and eigenfunctions of problem (29) are given by

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots. \tag{31}$$

As usual, the eigenfunctions $y_n(x)$ are determined only up to an arbitrary multiplicative constant. In the same way as for the problem (18), (19), you can show that the problem (29) has no eigenvalues or eigenfunctions other than those in equation (31).

CHAPTER 10

Partial Differential Equations and Fourier Series

10.2 Fourier Series

Later in this chapter you will find that you can solve many important problems involving partial differential equations, provided that you can express a given function as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**[2]; they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

We begin with a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

π

10.2 Fourier Series

Later in this chapter you will find that you can solve many important problems involving partial differential equations, provided that you can express a given function as an infinite sum of sines and/or cosines. In this and the following two sections we explain in detail how this can be done. These trigonometric series are called **Fourier series**[2]; they are somewhat analogous to Taylor series in that both types of series provide a means of expressing quite complicated functions in terms of certain familiar elementary functions.

We begin with a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \tag{1}$$

On the set of points where the series (1) converges, it defines a function f, whose value at each point is the sum of the series for that value of x. In this case the series (1) is said to be the **Fourier series** for f. Our immediate goals are to determine what functions can be represented as a sum of a Fourier series and to find some means of computing the coefficients in the series corresponding to a given function. The first term in the series (1) is written as $a_0/2$ rather than as a_0 to simplify a formula for the coefficients that we derive below. Besides their association with the method of separation of variables and partial differential equations, Fourier series are also useful in various other ways, such as in the analysis of mechanical or electrical systems acted on by periodic external forces.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

π

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$

Periodicity of the Sine and Cosine Functions. To discuss Fourier series, it is necessary to develop certain properties of the trigonometric functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer. The first property is their periodic character. A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x+T$ whenever it contains x, and if

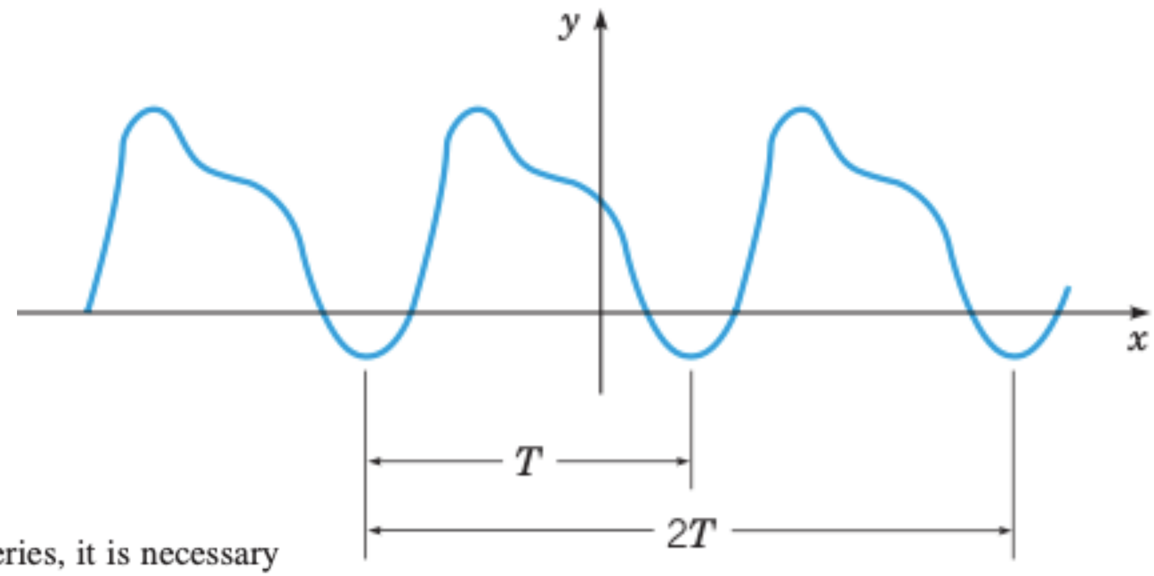
$$f(x+T) = f(x) \tag{2}$$

for every value of x. An example of a periodic function is shown in Figure 10.2.1. It follows immediately from the definition that if T is a period of f, then $2T$ is also a period, and so indeed is any integral multiple of T. The smallest value of T for which equation (2) holds is called the **fundamental period** of f. A constant function is a periodic function with an arbitrary period but no fundamental period.

10.2 Fourier Series

Partial Differential Equations and Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$



A periodic function of period T.

Periodicity of the Sine and Cosine Functions. To discuss Fourier series, it is necessary to develop certain properties of the trigonometric functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where m is a positive integer. The first property is their periodic character. A function f is said to be **periodic** with period $T > 0$ if the domain of f contains $x + T$ whenever it contains x, and if

$$f(x + T) = f(x) \tag{2}$$

for every value of x. An example of a periodic function is shown in Figure 10.2.1. It follows immediately from the definition that if T is a period of f, then $2T$ is also a period, and so indeed is any integral multiple of T. The smallest value of T for which equation (2) holds is called the **fundamental period** of f. A constant function is a periodic function with an arbitrary period but no fundamental period.

10.2 Fourier Series

$$\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\left(\frac{m\pi x}{L}\right)+b_m\sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

10.2 Fourier Series

π

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

If f and g are any two periodic functions with common period T, then any linear combination $c_1 f + c_2 g$ is also periodic with period T. To prove this statement, begin by defining $F(x) = c_1 f(x) + c_2 g(x)$; then, for any x,

$$F(x+T) = c_1 f(x+T) + c_2 g(x+T) = c_1 f(x) + c_2 g(x) = F(x). \qquad (3)$$

Moreover, it can be shown that the sum of any finite number, or even the sum of a convergent infinite series, of functions of period T is also periodic with period T. In a similar way, you can show that the product fg is periodic with period T.

In particular, the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, 3, \ldots$, are periodic with fundamental period $T = 2L/m$. To see this, recall that $\sin x$ and $\cos x$ have fundamental period 2π and that $\sin \alpha x$ and $\cos \alpha x$ have fundamental period $2\pi/\alpha$. If we choose $\alpha = m\pi/L$, then the period T of $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ is given by $T = 2\pi L/(m\pi) = 2L/m$.

Note also that, since every positive integral multiple of a period is also a period, each of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$ has the common period $2L$.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

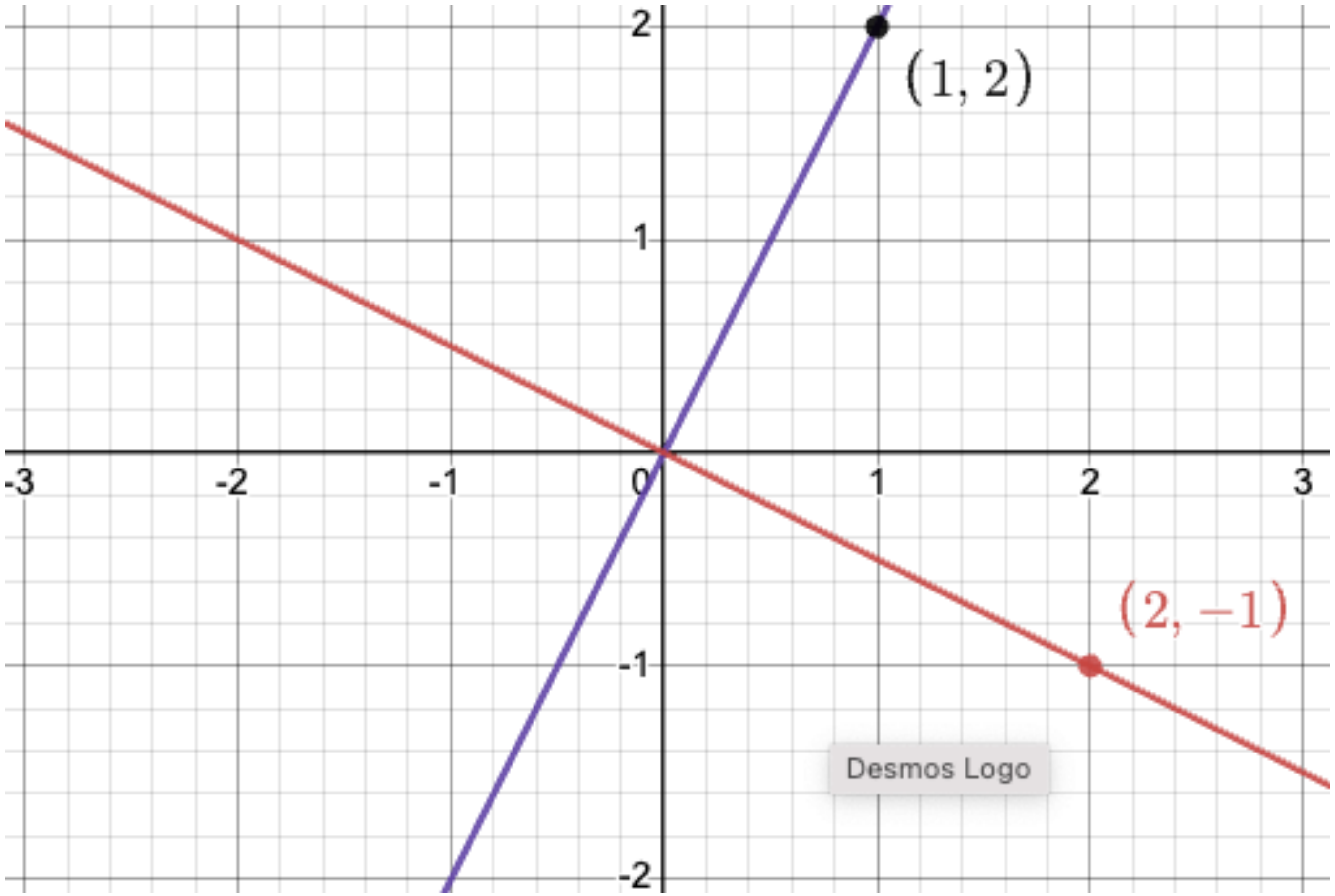
Orthogonal

π

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

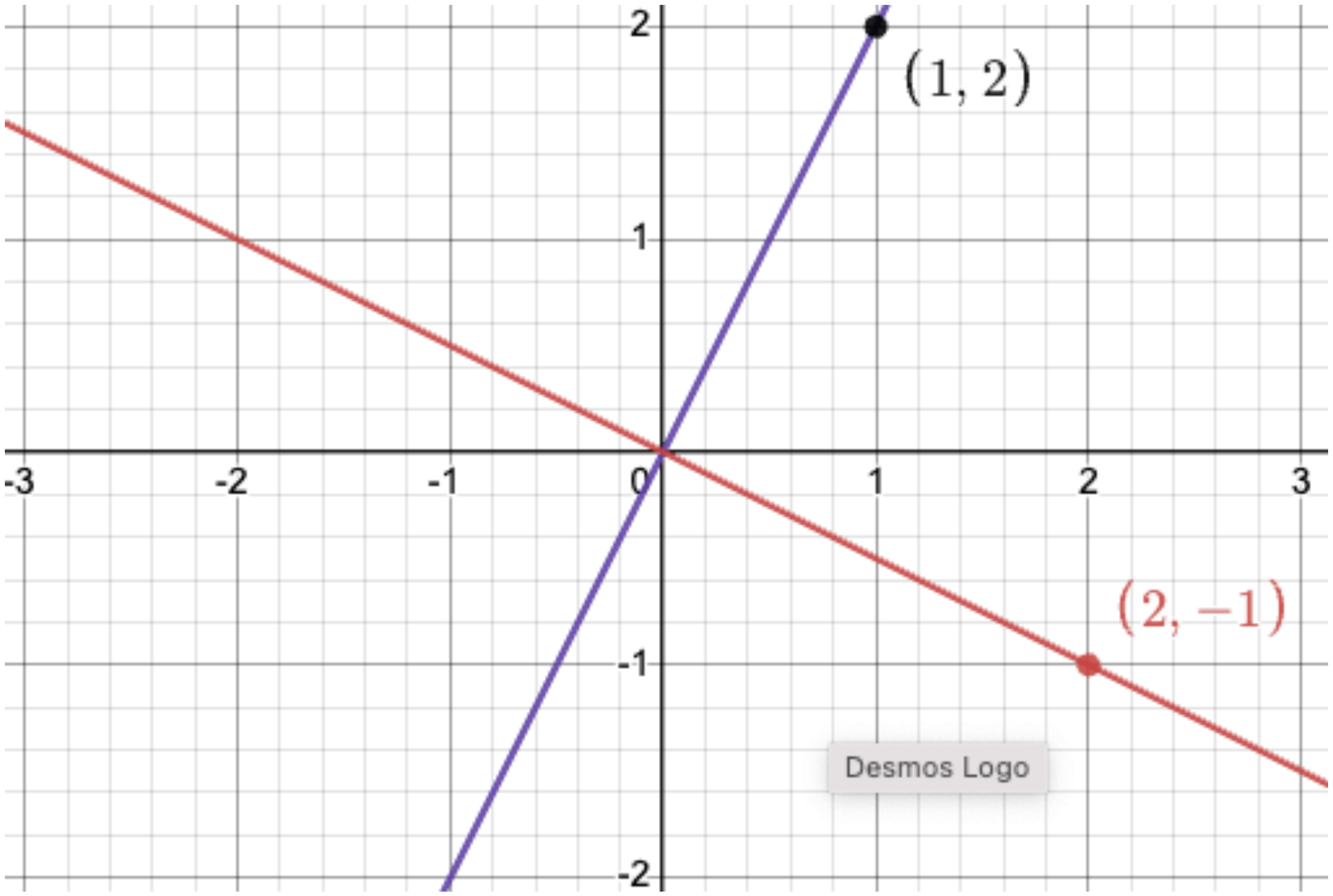
Orthogonal lines



10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$

Orthogonal vectors



π

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

Orthogonal vectors

Example 1:

Consider the following two vectors in 2D space:

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix},\ v_2 = \begin{bmatrix}2\\-1\end{bmatrix}$$

The dot product of these vectors is:

$$v_1 \cdot v_2 = v_1^T v_2 = (1 \times 2) + (2 \times -1) = 0$$

Because the dot product is zero, the angle between the vectors is 90° ($\cos 90^\circ = 0$).
Therefore, these two vectors are orthogonal.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

Orthogonality of the Sine and Cosine Functions. To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, we generalize the concept of orthogonality of vectors (see Section 7.2). The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx. \quad (4)$$

The functions u and v are said to be **orthogonal** on $\alpha \le x \le \beta$ if their inner product is zero—that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0. \quad (5)$$

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

10.2 Fourier Series

π

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$

Orthogonality of the Sine and Cosine Functions. To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, we generalize the concept of orthogonality of vectors (see Section 7.2). The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \le x \le \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx. \tag{4}$$

The functions u and v are said to be **orthogonal** on $\alpha \le x \le \beta$ if their inner product is zero—that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0. \tag{5}$$

A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$

The functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, \ldots$ form a mutually orthogonal set of functions on the interval $-L \le x \le L$. In fact, they satisfy the following orthogonality relations:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n; \end{cases} \tag{6}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{all } m, n; \tag{7}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases} \tag{8}$$

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

These results can be obtained by direct integration. For example, to derive equation (8), note that

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right)dx$$

$$= \frac{1}{2}\frac{L}{\pi}\left(\frac{\sin((m-n)\pi x/L)}{m-n} - \frac{\sin((m+n)\pi x/L)}{m+n}\right)\Bigg|_{-L}^{L}$$

$$= 0$$

as long as $m+n$ and $m-n$ are not zero. Since m and n are positive, $m+n \neq 0$. On the other hand, if $m-n=0$, then $m=n$, and the integral must be evaluated in a different way. In this case:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L}\left(\sin\left(\frac{m\pi x}{L}\right)\right)^2 dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\left(\frac{2m\pi x}{L}\right)\right)dx$$

$$= \frac{1}{2}\left(x - \frac{L}{2m\pi}\sin\left(\frac{2m\pi x}{L}\right)\right)\Bigg|_{-L}^{L}$$

$$= L.$$

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \qquad (1)$$

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (1)$$

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \leq x \leq L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \qquad (9)$$

π

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \le x \le L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (9)$$

Multiply through by orthogonal functions...

π

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (1)$$

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \leq x \leq L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \quad (9)$$

Multiply through by orthogonal functions...

Term by term integration...

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{1}$$

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \le x \le L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right). \tag{9}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots . \tag{13}$$

By writing the constant term in equation (9) as $a_0/2$, it is possible to compute all the a_n from equation (13). Otherwise, a separate formula, with an extra factor of $1/2$, would have to be used for a_0.

10.2 Fourier Series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \tag{1}$$

The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges for all real numbers x on the interval $-L \le x \le L$, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right). \tag{9}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots . \tag{13}$$

By writing the constant term in equation (9) as $a_0/2$, it is possible to compute all the a_n from equation (13). Otherwise, a separate formula, with an extra factor of $1/2$, would have to be used for a_0.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots . \tag{14}$$

10.2 Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots . \tag{13}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots . \tag{14}$$

10.2 Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots . \tag{13}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots . \tag{14}$$

EXAMPLE 1

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases}$$
$$f(x+4) = f(x). \tag{15}$$

Determine the coefficients in this Fourier series.

10.2 Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

EXAMPLE 1

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x). \tag{15}$$

Determine the coefficients in this Fourier series.

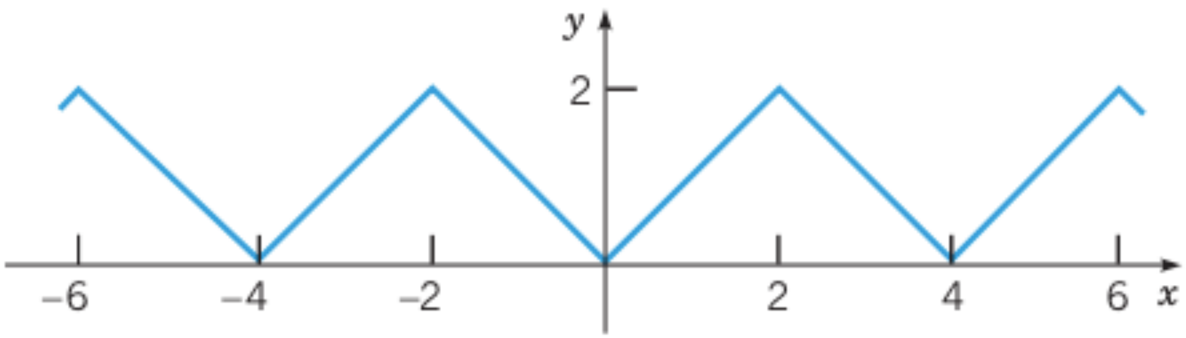


FIGURE 10.2.2 The triangular wave in Example 1.

10.2 Fourier Series

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

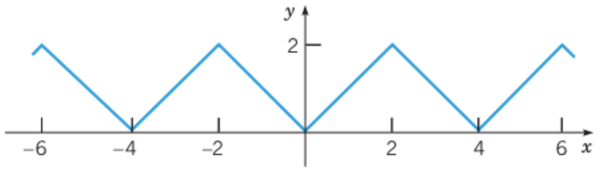


FIGURE 10.2.2 The triangular wave in Example 1.

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad (15)$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos\left(\frac{m\pi x}{2}\right) + b_m\sin\left(\frac{m\pi x}{2}\right)\right), \qquad (16)$$

where the coefficients are computed from equations (13) and (14) with $L = 2$. Substituting for $f(x)$ in equation (13) with $m = 0$, we have

$$a_0 = \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 1 + 1 = 2. \qquad (17)$$

For $m > 0$, equation (13) yields

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\left(\frac{m\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{m\pi x}{2}\right)dx.$$

Partial Differential Equations and Fourier Series

10.2 Fourier Series

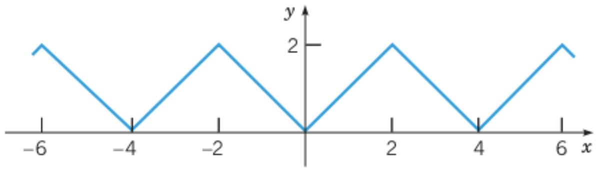


FIGURE 10.2.2 The triangular wave in Example 1.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \dots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \dots$$

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad (15)$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\left(\frac{m\pi x}{2}\right)dx$$

$$\int u \cdot dv = uv - \int v \cdot du$$

where: $u = -x$

$$dv = \cos\left(\frac{m\pi x}{2}\right)$$

$$\implies \frac{1}{2}\int_{-2}^{0} -x\cos\left(\frac{m\pi x}{2}\right)dx = \left(-\frac{2}{m\pi}x\sin\left(\frac{m\pi x}{2}\right)\right) - \dots$$

$$-\frac{1}{2}\left(\int_{-2}^{0} -\sin\left(\frac{m\pi x}{2}\right)dx\right).$$

10.2 Fourier Series

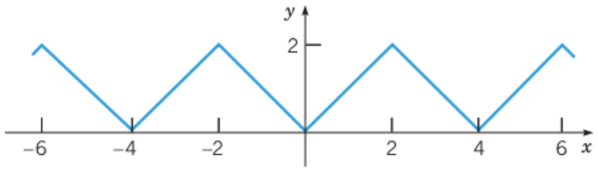


FIGURE 10.2.2 The triangular wave in Example 1.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots .$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad (15)$$

$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

$$a_m = \frac{1}{2}\int_{-2}^{0} (-x)\cos\left(\frac{m\pi x}{2}\right)dx$$

$$\int u \cdot dv = uv - \int v \cdot du$$

where: $u = -x$

$$dv = \cos\left(\frac{m\pi x}{2}\right)$$

$$\implies \frac{1}{2}\int_{-2}^{0} -x\cos\left(\frac{m\pi x}{2}\right)dx = \left(-\frac{2}{m\pi}x\sin\left(\frac{m\pi x}{2}\right)\right) - \ldots$$

$$-\frac{1}{2}\left(\int_{-2}^{0}\sin\left(\frac{m\pi x}{2}\right)dx\right).$$

10.2 Fourier Series

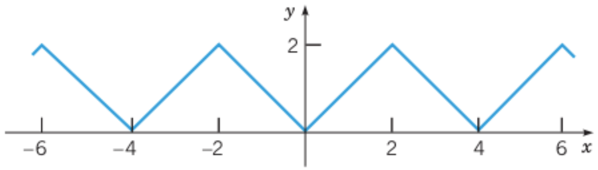


FIGURE 10.2.2 The triangular wave in Example 1.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots .$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad (15)$$
$$f(x+4) = f(x).$$

Determine the coefficients in this Fourier series.

These integrals can be evaluated through integration by parts, with the result that

$$a_m = \frac{1}{2}\left(-\frac{2}{m\pi}x\sin\left(\frac{m\pi x}{2}\right) - \left(\frac{2}{m\pi}\right)^2\cos\left(\frac{m\pi x}{2}\right)\right)\Bigg|_{-2}^{0}$$
$$+\frac{1}{2}\left(\frac{2}{m\pi}x\sin\left(\frac{m\pi x}{2}\right) + \left(\frac{2}{m\pi}\right)^2\cos\left(\frac{m\pi x}{2}\right)\right)\Bigg|_{0}^{2}$$
$$= \frac{1}{2}\left(-\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2\cos(m\pi) + \left(\frac{2}{m\pi}\right)^2\cos(m\pi) - \left(\frac{2}{m\pi}\right)^2\right)$$
$$= \frac{4}{(m\pi)^2}(\cos(m\pi) - 1), \quad m = 1, 2, \ldots$$
$$= \begin{cases} -\dfrac{8}{(m\pi)^2}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases}$$

10.2 Fourier Series

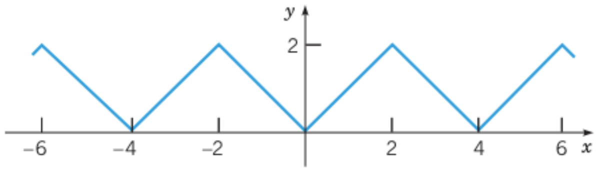


FIGURE 10.2.2 The triangular wave in Example 1.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots$$

EXAMPLE 1

Assume that there is a Fourier series converging to the function f defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x). \tag{15}$$

Determine the coefficients in this Fourier series.

Finally, from equation (14), it follows in a similar way that

$$b_m = 0, \quad m = 1, 2, \ldots. \tag{19}$$

By substituting the coefficients from equations (17), (18), and (19) in the series (16), we obtain the Fourier series for f:

$$\begin{aligned} f(x) &= 1 - \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{2}\right) + \cdots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m^2}\cos\left(\frac{m\pi x}{2}\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{2}\right). \end{aligned} \tag{20}$$

π

10.4 Even and Odd Functions

Before looking at further examples of Fourier series, it is useful to distinguish two classes of functions for which the Euler-Fourier formulas can be simplified. These are even and odd functions, which are characterized geometrically by the property of symmetry with respect to the y-axis and with respect to the origin, respectively (see Figure 10.4.1).

10.4 Even and Odd Functions

10.4 Even and Odd Functions

$$\text{Even number} := 2 \cdot k, \ \forall k \ \in \mathbb{Z},$$
$$\text{Odd number} := 2 \cdot k + 1, \ \forall k \ \in \mathbb{Z}.$$

10.4 Even and Odd Functions

Rules for Even and Odd Numbers

Addition	*Examples*	Subtraction	*Examples*
even + even = even odd + odd = even even + odd = odd odd + even = odd	2 + 2 = 4 1 + 1 = 2 2 + 1 = 3 1 + 2 = 3	even - even = even odd - odd = even even - odd = odd odd - even = odd	4 - 2 = 2 1 - 3 = -2 2 - 3 = -1 3 - 2 = 1
Multiplication	*Examples*	**Division**	*Examples*
even × even = even odd × odd = odd even × odd = even odd × even = even	2 × 2 = 4 1 × 1 = 1 2 × 1 = 2 1 × 2 = 2	*(only true if the quotient is a whole number)* even ÷ odd = even odd ÷ odd = odd even ÷ even = odd or even ÷ even = even odd ÷ even = not a whole number	6 ÷ 3 = 2 9 ÷ 3 = 3 6 ÷ 2 = 3 4 ÷ 2 = 2

$$\text{Even number} := 2 \cdot k, \ \forall k \in \mathbb{Z},$$

$$\text{Odd number} := 2 \cdot k + 1, \ \forall k \in \mathbb{Z}.$$

10.4 Even and Odd Functions

10.4 Even and Odd Functions

Analytically, f is an **even function** if its domain contains the point $-x$ whenever it contains the point x, and if

$$f(-x) = f(x) \tag{1}$$

for each x in the domain of f. Similarly, f is an **odd function** if its domain contains $-x$ whenever it contains x, and if

$$f(-x) = -f(x) \tag{2}$$

for each x in the domain of f. Examples of even functions are 1, x^2, $\cos(nx)$, $|x|$, $\cosh(nx)$, and x^{2n}. The functions x, x^3, $\sin(nx)$, $\sinh(nx)$, and x^{2n+1} are examples of odd functions. Note that according to equation (2), $f(0)$ must be zero if f is an odd function whose domain contains the origin. Most functions are neither even nor odd; an example is e^x. Only one function, f identically zero, is both even and odd.

π

10.4 Even and Odd Functions

10.4 Even and Odd Functions

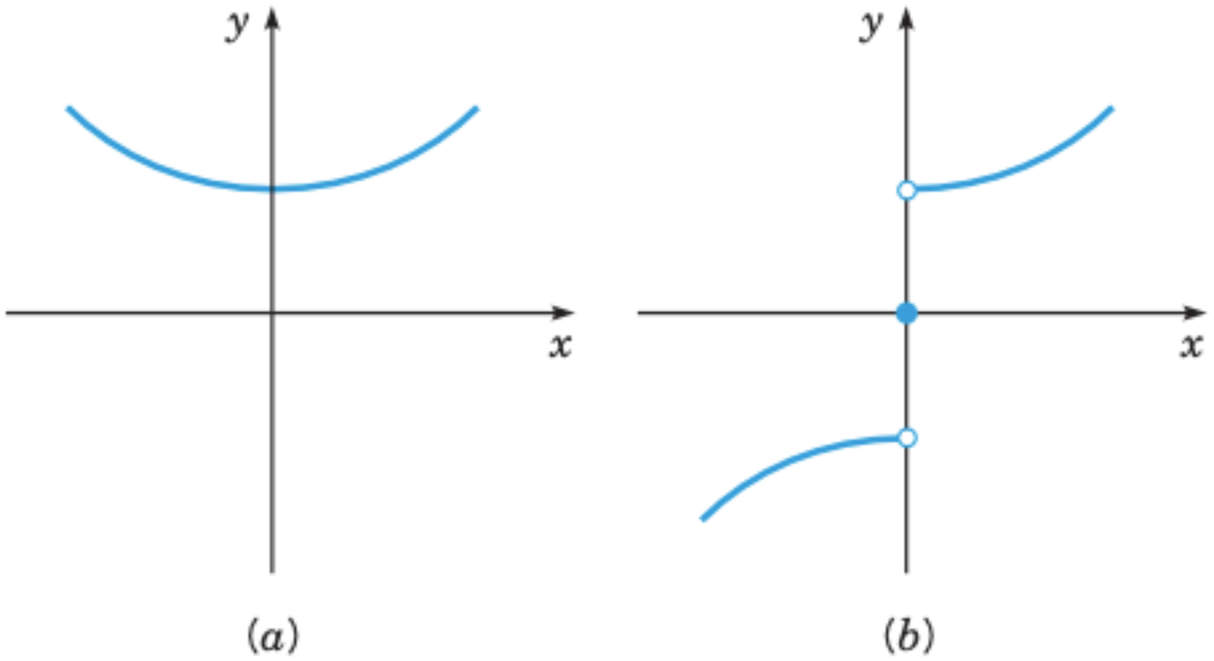


FIGURE 10.4.1 (*a*) An even function. (*b*) An odd function.

Partial Differential Equations and Fourier Series

10.4 Even and Odd Functions

π

10.4 Even and Odd Functions

Elementary properties of even and odd functions include the following:[8]

1. The sum (difference) and product (quotient) of two even functions are even.
2. The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even.
3. The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of an odd function and an even function is odd.

π

10.4 Even and Odd Functions

Elementary properties of even and odd functions include the following:[8]

1. The sum (difference) and product (quotient) of two even functions are even.
2. The sum (difference) of two odd functions is odd; the product (quotient) of two odd functions is even.
3. The sum (difference) of an odd function and an even function is neither even nor odd; the product (quotient) of an odd function and an even function is odd.

The proofs of all these assertions are simple and follow directly from the definitions. For example, if both f_1 and f_2 are odd, and if $g(x) = f_1(x) + f_2(x)$, then

$$\begin{aligned} g(-x) &= f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) \\ &= -\big(f_1(x) + f_2(x)\big) = -g(x), \end{aligned} \tag{3}$$

so $f_1 + f_2$ is an odd function also. Similarly, if $h(x) = f_1(x) f_2(x)$, then

$$h(-x) = f_1(-x) f_2(-x) = \big(-f_1(x)\big)\big(-f_2(x)\big) = f_1(x) f_2(x) = h(x), \tag{4}$$

so that $f_1 f_2$ is even.

10.4 Even and Odd Functions

10.4 Even and Odd Functions

π

4. If f is an even function, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx. \tag{5}$$

5. If f is an odd function, then

$$\int_{-L}^{L} f(x)\,dx = 0. \tag{6}$$

10.4 Even and Odd Functions

10.4 Even and Odd Functions

Cosine Series. Suppose that f and f' are piecewise continuous on $-L \leq x < L$ and that f is an even periodic function with period $2L$. Then it follows from properties 1 and 3 that $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd. As a consequence of equations (5) and (6), the Fourier coefficients of f are then given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots;$$
$$b_n = 0, \quad n = 1, 2, \ldots. \tag{7}$$

Thus f has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

In other words, the Fourier series of any even function consists only of the even trigonometric functions $\cos(n\pi x/L)$ and the constant term; it is natural to call such a series a **Fourier cosine series**. From a computational point of view, observe that only the coefficients a_n, for $n = 0, 1, 2, \ldots$, need to be calculated from the integral formula (7). Each of the b_n, for $n = 1, 2, \ldots$, is automatically zero for any even function and so does not need to be calculated by integration.

10.4 Even and Odd Functions

Sine Series. Suppose that f and f' are piecewise continuous on $-L \leq x < L$ and that f is an odd periodic function of period $2L$. Then it follows from Properties 2 and 3 that $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even. Thus, from equations (5) and (6), the Fourier coefficients of f are

$$a_n = 0, \qquad n = 0, 1, 2, \dots,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \dots, \tag{8}$$

and the Fourier series for f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Thus the Fourier series for any odd function consists only of the odd trigonometric functions $\sin(n\pi x/L)$; such a series is called a **Fourier sine series**. Again observe that only half of the coefficients need to be calculated by integration, since each a_n, for $n = 0, 1, 2, \dots,$ is zero for any odd function.

10.4 Even and Odd Functions

10.4 Even and Odd Functions

Example 1:

Let $f(x) = x$, $-L < x < L$, and let $f(-L) = f(L) = 0$. Let f be defined elsewhere so that it is periodic of period $2L$. The function defined in this manner is known as a *sawtooth wave*. Graph three periods of $y = f(x)$. Find the Fourier series for this function.

ntial
d

Solution:

The graph of $y = f(x)$ on $[-L, L]$ and one period to the left and one period to the right is shown in Figure 10.4.2.

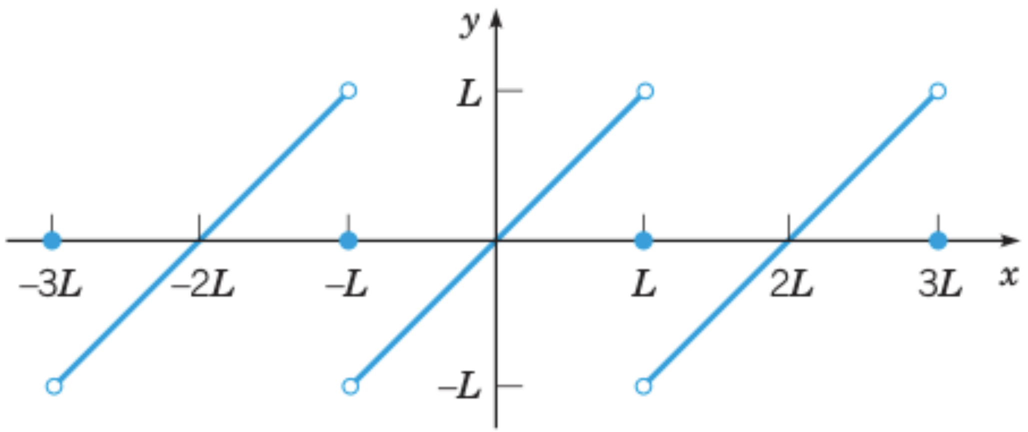


FIGURE 10.4.2 The sawtooth wave in Example 1.

Since f is an odd function, its Fourier coefficients are, according to equation (8),

$$a_n = 0, \quad n = 0, 1, 2, \ldots;$$

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left(\sin\left(\frac{n\pi x}{L}\right) - \frac{n\pi x}{L}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L$$

$$= \frac{2L}{n\pi}(-1)^{n+1}, \quad n = 1, 2, \ldots.$$

Solution:

The graph of $y = f(x)$ on $[-L, L]$ and one period to the left and one period to the right is shown in Figure 10.4.2.

ntial

d

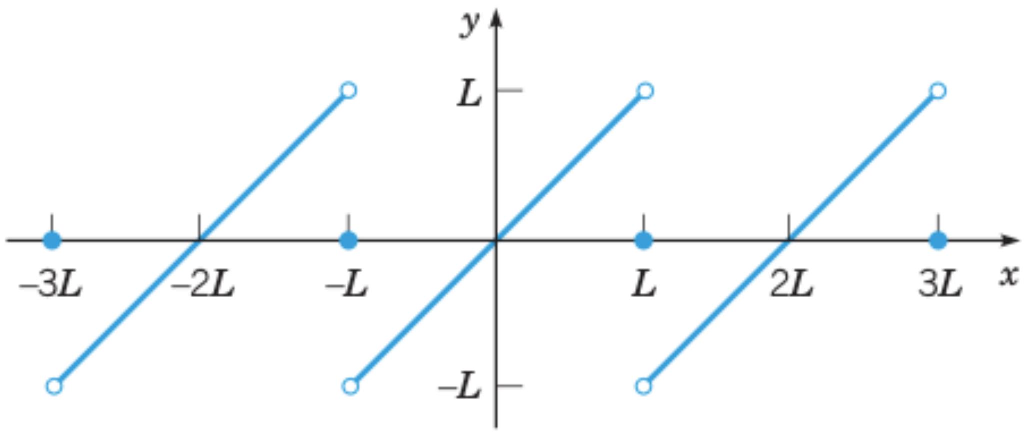


FIGURE 10.4.2 The sawtooth wave in Example 1.

Hence the Fourier series for f, the sawtooth wave, is

$$\begin{aligned} f(x) &= \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2L}{\pi} \left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5} \sin\left(\frac{5\pi x}{L}\right) - \cdots \right). \end{aligned} \tag{9}$$

10.4 Even and Odd Functions

10.4 Even and Odd Functions

In solving problems in differential equations, it is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on the interval $[0, L]$. As indicated previously for the function $f(x) = x$, with $L = 2$, several alternatives are available. Explicitly, we can

1. Define a function g of period $2L$ so that

$$g(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ f(-x), & -L < x < 0. \end{cases} \tag{10}$$

The function g is thus the *even periodic extension* of f. Its Fourier series, which is a cosine series, represents f on $[0, L]$.

10.4 Even and Odd Functions

In solving problems in differential equations, it is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on the interval $[0, L]$. As indicated previously for the function $f(x) = x$, with $L = 2$, several alternatives are available. Explicitly, we can

2. Define a function h of period $2L$ so that

$$h(x) = \begin{cases} f(x), & 0 < x < L, \\ 0, & x = 0, L, \\ -f(-x), & -L < x < 0. \end{cases} \tag{11}$$

The function h is thus the *odd periodic extension* of f. Its Fourier series, which is a sine series, also represents f on $(0, L)$.

10.4 Even and Odd Functions

In solving problems in differential equations, it is often useful to expand in a Fourier series of period $2L$ a function f originally defined only on the interval $[0, L]$. As indicated previously for the function $f(x) = x$, with $L = 2$, several alternatives are available. Explicitly, we can

3. Define a function k of period $2L$ so that

$$k(x) = f(x), \quad 0 \leq x \leq L, \tag{12}$$

and let $k(x)$ be defined for $(-L, 0)$ in any way consistent with the conditions of Theorem 10.3.1. Sometimes it is convenient to define $k(x)$ to be zero for $-L < x < 0$. The Fourier series for k, which involves both sine and cosine terms, also represents f on $[0, L]$, regardless of the manner in which $k(x)$ is defined in $(-L, 0)$. Thus there are infinitely many such series, all of which converge to $f(x)$ in the original interval.

π

10.4 Even and Odd Functions

Suppose that

$$f(x) = \begin{cases} 1 - x, & 0 < x \le 1, \\ 0, & 1 < x \le 2. \end{cases} \tag{13}$$

As indicated previously, we can represent f either by a cosine series or by a sine series. Sketch the graph of three periods of the sum of each of these series for $-6 \le x \le 6$.

Solution:

In this example, $L = 2$, so the cosine series for f converges to the even periodic extension of f of period 4, whose graph is sketched in Figure 10.4.4.

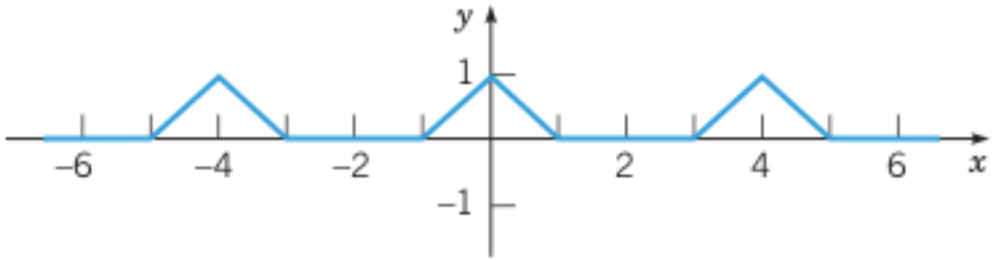


FIGURE 10.4.4 Even periodic extension of $f(x)$ given by equation (13).

Similarly, the sine series for f converges to the odd periodic extension of f of period 4. The graph of this function is shown in Figure 10.4.5.

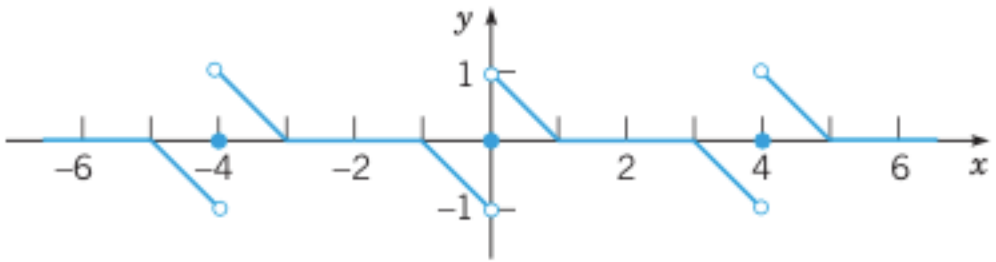


FIGURE 10.4.5 Odd periodic extension of $f(x)$ given by equation (13).

π

10.4 Even and Odd Functions

Study hard, good luck, meet your academic goals.

π