# MAT112 -Mr. José Pabón

*Recitation will start soon.*

We will pass this course with a great grade & meet our academic and professional goals!

- MAT112 T.A. Mr. José Pabón

We will be courteous, civil to each other.
NO SUCH THING AS AN OBVIOUS QUESTION
ask ask ask any doubt to clear up

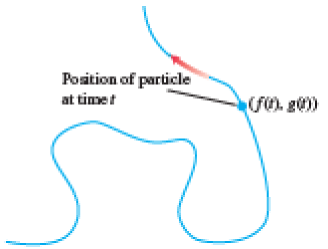


**FIGURE 11.1** The curve or path traced by a particle moving in the $xy$-plane is not always the graph of a function or single equation.

**DEFINITION** If $x$ and $y$ are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval $I$ of $t$-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

**EXAMPLE 2** Sketch the curve defined by the parametric equations

$$x = t^2, \qquad y = t + 1, \qquad -\infty < t < \infty.$$

**Solution** We make a table of values (Table 11.2), plot the points $(x, y)$, and draw a smooth curve through them (Figure 11.3). We think of the curve as the path that a particle moves along the curve in the direction of the arrows. Although the time intervals in the table are equal, the consecutive points plotted along the curve are not at equal arc length distances. The reason for this is that the particle slows down as it gets nearer to the $y$-axis along the lower branch of the curve as $t$ increases, and then speeds up after reaching the $y$-axis at (0, 1) and moving along the upper branch. Since the interval of values for $t$ is all real numbers, there is no initial point and no terminal point for the curve.

**TABLE 11.2** Values of $x = t^2$ and $y = t + 1$ for selected values of $t$.

| $t$ | $x$ | $y$ |
|---|---|---|
| −3 | 9 | −2 |
| −2 | 4 | −1 |
| −1 | 1 | 0 |
| 0 | 0 | 1 |
| 1 | 1 | 2 |
| 2 | 4 | 3 |
| 3 | 9 | 4 |

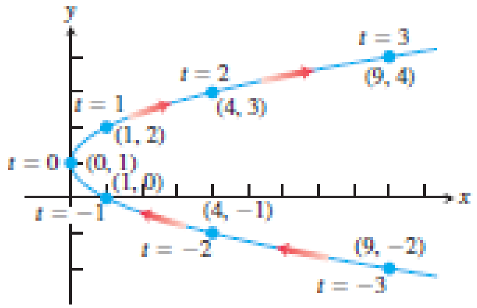


**FIGURE 11.3** The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 2).

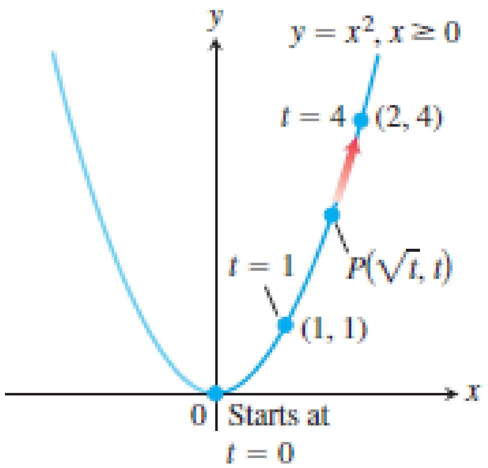


**FIGURE 11.5** The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the path of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 4).

**EXAMPLE 4** The position $P(x, y)$ of a particle moving in the $xy$-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

**Solution** We try to identify the path by eliminating $t$ between the equations $x = \sqrt{t}$ and $y = t$, which might produce a re-cognizable algebraic relation between $x$ and $y$. We find that

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

It would be a mistake, however, to conclude that the particle's path is the entire parabola $y = x^2$; it is only half the parabola. The particle's $x$-coordinate is never negative. The particle starts at (0, 0) when $t = 0$ and rises into the first quadrant as $t$ increases (Figure 11.5). The parameter interval is $[0, \infty)$ and there is no terminal point. ■

**TABLE 11.3** Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of $t$.

| $t$ | $1/t$ | $x$ | $y$ |
|---|---|---|---|
| 0.1 | 10.0 | 10.1 | −9.9 |
| 0.2 | 5.0 | 5.2 | −4.8 |
| 0.4 | 2.5 | 2.9 | −2.1 |
| 1.0 | 1.0 | 2.0 | 0.0 |
| 2.0 | 0.5 | 2.5 | 1.5 |
| 5.0 | 0.2 | 5.2 | 4.8 |
| 10.0 | 0.1 | 10.1 | 9.9 |

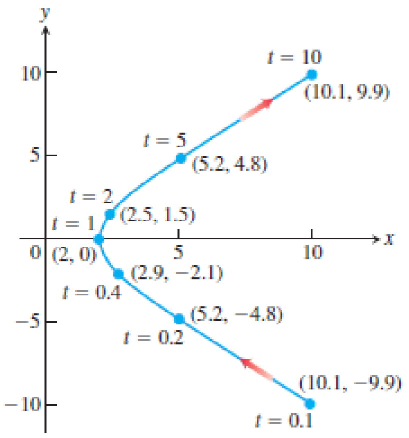


**FIGURE 11.7** The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \le t \le 10$.)

**EXAMPLE 7** Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

**Solution** We make a brief table of values in Table 11.3, plot the points, and draw a smooth curve through them, as we did in Example 1. Next we eliminate the parameter $t$ from the equations. The procedure is more complicated than in Example 2. Taking the difference between $x$ and $y$ as given by the parametric equations, we find that

$$x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}.$$

If we add the two parametric equations, we get

$$x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

We can then eliminate the parameter $t$ by multiplying these last equations together:

$$(x - y)(x + y) = \left(\frac{2}{t}\right)(2t) = 4.$$

Expanding the expression on the left-hand side, we obtain a standard equation for a hyperbola (reviewed in Section 11.6):

$$x^2 - y^2 = 4. \qquad (1)$$

Thus the coordinates of all the points $P(x, y)$ described by the parametric equations satisfy Equation (1). However, Equation (1) does not require that the $x$-coordinate be positive. So there are points $(x, y)$ on the hyperbola that do not satisfy the parametric equation $x = t + (1/t)$, $t > 0$. In fact, the parametric equations do not yield any points on the left branch of the hyperbola given by Equation (1), points where the $x$-coordinate would be negative. For small positive values of $t$, the path lies in the fourth quadrant and rises into the first quadrant as $t$ increases, crossing the $x$-axis when $t = 1$ (see Figure 11.7). The parameter domain is $(0, \infty)$ and there is no starting point and no terminal point for the path. ∎

## Finding Cartesian from Parametric Equations

Exercises 1–18 give parametric equations and parameter intervals for the motion of a particle in the $xy$-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

**13.** $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \le t \le 0$

**14.** $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \ge 0$

**15.** $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

**16.** $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

**17.** $x = -\cosh t, \quad y = \sinh t, \quad -\infty < t < \infty$

**18.** $x = 2 \sinh t, \quad y = 2 \cosh t, \quad -\infty < t < \infty$

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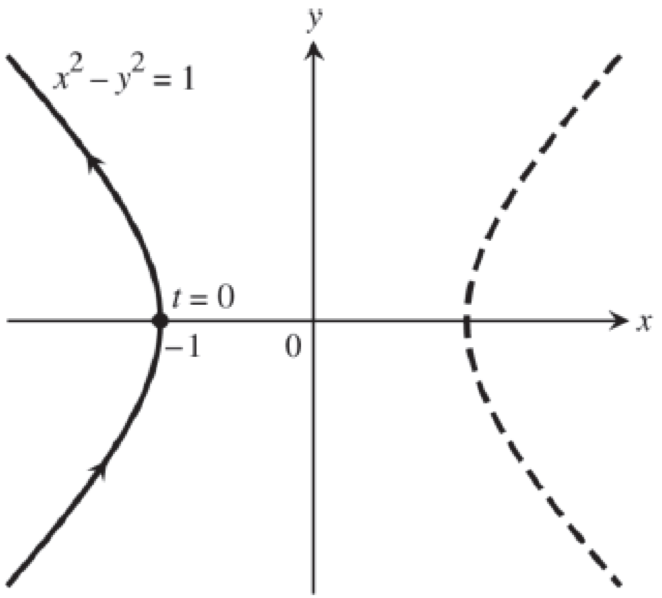
17. $x = -\cosh t,\ y = \sinh t,\ -\infty < 1 < \infty$

$\Rightarrow \cosh^2 t - \sinh^2 t = 1 \Rightarrow x^2 - y^2 = 1$



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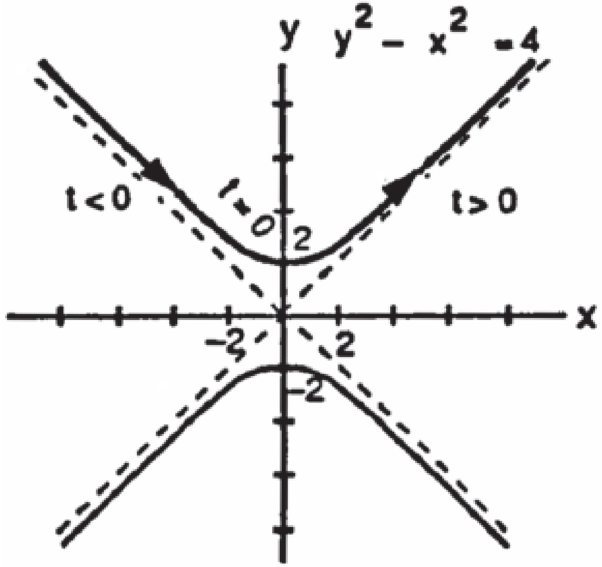
18. $x = 2\sinh t,\ y = 2\cosh t,\ -\infty < t < \infty$

$\Rightarrow 4\cosh^2 t - 4\sinh^2 t = 4 \Rightarrow y^2 - x^2 = 4$



### Distance Using Parametric Equations

**49.** Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of $t$.)

### Distance Using Parametric Equations

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49. $D = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} \Rightarrow D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (t-2)^2 + \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$

## Distance Using Parametric Equations

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49. $D = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} \Rightarrow D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (t-2)^2 + \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$

$\Rightarrow \frac{d\left(D^2\right)}{dt} = 4t^3 - 4 = 0 \Rightarrow t = 1$. The second derivative is always positive for $t \neq 0 \Rightarrow t = 1$ gives a local minimum for $D^2$ (and hence $D$) which is an absolute minimum since it is the only extremum $\Rightarrow$ the closest point on the parabola is $(1, 1)$.

## Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at $t$ if $f$ and $g$ are differentiable at $t$. At a point on a differentiable parametrized curve where $y$ is also a differentiable function of $x$, the derivatives $dy/dt$, $dx/dt$, and $dy/dx$ are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by $dx/dt$ to solve for $dy/dx$.

### Parametric Formula for $dy/dx$

If all three derivatives exist and $dx/dt \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

If parametric equations define $y$ as a twice-differentiable function of $x$, we can apply Equation (1) to the function $dy/dx = y'$ to calculate $d^2y/dx^2$ as a function of $t$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$ Eq. (1) with $y'$ in place of $y$

### Parametric Formula for $d^2y/dx^2$

If the equations $x = f(t)$, $y = g(t)$ define $y$ as a twice-differentiable function of $x$, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$

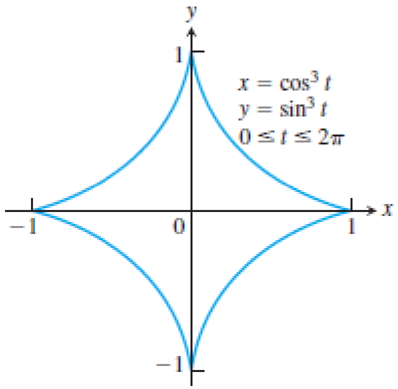


**FIGURE 11.15** The astroid in Example 3.

**EXAMPLE 3** Find the area enclosed by the astroid (Figure 11.15)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

**Solution** By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0 \le t \le \pi/2$. We can apply the definite integral formula for area studied in Chapter 5, using substitution to express the curve and differential $dx$ in terms of the parameter $t$. Thus,

$$A = 4\int_0^1 y\,dx \qquad \text{4 times area under } y \text{ from } x = 0 \text{ to } x = 1$$

$$= 4\int_0^{\pi/2} (\sin^3 t)(3\cos^2 t \sin t)\,dt \qquad \text{Substitution for } y \text{ and } dx$$

$$= 12\int_0^{\pi/2} \left(\frac{1 - \cos 2t}{2}\right)^2 \left(\frac{1 + \cos 2t}{2}\right) dt \qquad \sin^4 t = \left(\frac{1 - \cos 2t}{2}\right)^2$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - 2\cos 2t + \cos^2 2t)(1 + \cos 2t)\,dt \qquad \text{Expand squared term.}$$

$$= \frac{3}{2}\int_0^{\pi/2} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\,dt \qquad \text{Multiply terms.}$$

$$= \frac{3}{2}\left[\int_0^{\pi/2}(1 - \cos 2t)\,dt - \int_0^{\pi/2}\cos^2 2t\,dt + \int_0^{\pi/2}\cos^3 2t\,dt\right]$$

$$= \frac{3}{2}\left[\left(t - \frac{1}{2}\sin 2t\right) - \frac{1}{2}\left(t + \frac{1}{4}\sin 2t\right) + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2}$$

Section 8.2, Example 3

$$= \frac{3}{2}\left[\left(\frac{\pi}{2} - 0 - 0 - 0\right) - \frac{1}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) + \frac{1}{2}(0 - 0 - 0 + 0)\right]$$

Evaluate.

$$= \frac{3\pi}{8}.$$

■

**DEFINITION** If a curve $C$ is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where $f'$ and $g'$ are continuous and not simultaneously zero on $[a, b]$, and $C$ is traversed exactly once as $t$ increases from $t = a$ to $t = b$, then **the length of $C$** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

**EXAMPLE 4** Using the definition, find the length of the circle of radius $r$ defined parametrically by

$$x = r\cos t \quad \text{and} \quad y = r\sin t, \quad 0 \leq t \leq 2\pi.$$

**Solution** As $t$ varies from 0 to $2\pi$, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find

$$\frac{dx}{dt} = -r\sin t, \qquad \frac{dy}{dt} = r\cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

Therefore, the total arc length is

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\Big[t\Big]_0^{2\pi} = 2\pi r. \quad \blacksquare$$

### Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

**1. Revolution about the $x$-axis ($y \ge 0$):**

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (5)$$

**2. Revolution about the $y$-axis ($x \ge 0$):**

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (6)$$

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

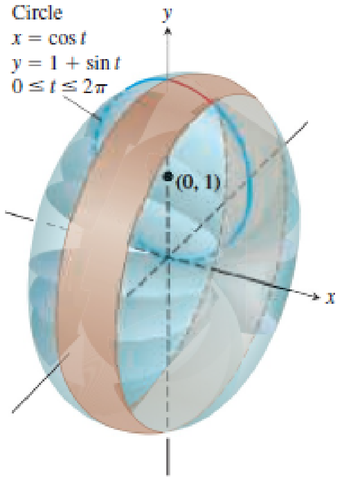


**FIGURE 11.19** In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

**EXAMPLE 9** The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the $xy$-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the $x$-axis (Figure 11.19).

**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Eq. (5) for revolution about the $x$-axis; $y = 1 + \sin t \ge 0$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi \Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$

■

### Tangents to Parametrized Curves

In Exercises 1–14, find an equation for the line tangent to the curve at the point defined by the given value of $t$. Also, find the value of $d^2y/dx^2$ at this point.

1. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$
2. $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -1/6$
3. $x = 4\sin t, \quad y = 2\cos t, \quad t = \pi/4$
4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$
7. $x = \sec t, \quad y = \tan t, \quad t = \pi/6$
8. $x = -\sqrt{t+1}, \quad y = \sqrt{3t}, \quad t = 3$
9. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$
10. $x = 1/t, \quad y = -2 + \ln t, \quad t = 1$
11. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$
12. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$
13. $x = \dfrac{1}{t+1}, \quad y = \dfrac{t}{t-1}, \quad t = 2$
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

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13. $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

12. $t = \frac{\pi}{2} \Rightarrow x = \cos\frac{\pi}{2} = 0, \quad y = 1 + \sin\frac{\pi}{2} = 2; \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -\cot\frac{\pi}{2} = 0; \text{ tangent line is } y = 2; \quad \frac{dy'}{dt} = \csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t} = -\csc^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -1$$

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4. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$
5. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$
6. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$
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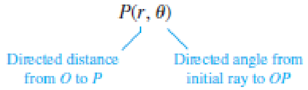
14. $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

13. $t = 2 \Rightarrow x = \frac{1}{2+1} = \frac{1}{3}, \quad y = \frac{2}{2-1} = 2; \quad \frac{dx}{dt} = \frac{-1}{(t+1)^2}, \quad \frac{dy}{dt} = \frac{-1}{(t-1)^2} \Rightarrow \frac{dy}{dx} = \frac{(t+1)^2}{(t-1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{(2+1)^2}{(2-1)^2} = 9;$ tangent line is

$$y = 9x - 1; \quad \frac{dy'}{dt} = -\frac{4(t+1)}{(t-1)^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{4(t+1)^3}{(t-1)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{4(2+1)^3}{(2-1)^3} = 108$$

## Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** $O$ (called the **pole**) and an **initial ray** from $O$ (Figure 11.20). Usually the positive $x$-axis is chosen as the initial ray. Then each point $P$ can be located by assigning to it a **polar coordinate pair** $(r, \theta)$ in which $r$ gives the directed distance from $O$ to $P$ and $\theta$ gives the directed angle from the initial ray to ray $OP$. So we label the point $P$ as




As in trigonometry, $\theta$ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Figure 11.21). In some situations we allow $r$ to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 11.22). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.

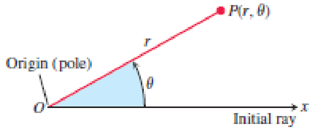



**FIGURE 11.20** To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

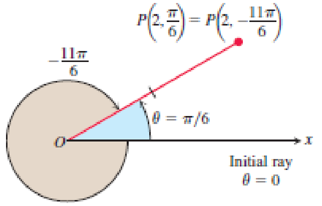



**FIGURE 11.21** Polar coordinates are not unique.

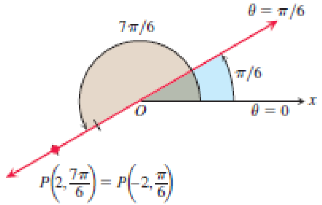



**FIGURE 11.22** Polar coordinates can have negative $r$-values.

**EXAMPLE 1** Find all the polar coordinates of the point $P(2, \pi/6)$.

**Solution** We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.23). We then find the angles for the other coordinate pairs of $P$ in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots .$$

The corresponding coordinate pairs of $P$ are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

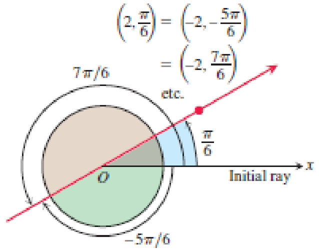



**FIGURE 11.23** The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

## Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and let the initial polar ray be the positive $x$-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive $y$-axis (Figure 11.26). The two coordinate systems are then related by the following equations.

> **Equations Relating Polar and Cartesian Coordinates**
>
> $$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

The first two of these equations uniquely determine the Cartesian coordinates $x$ and $y$ given the polar coordinates $r$ and $\theta$. On the other hand, if $x$ and $y$ are given, the third equation gives two possible choices for $r$ (a positive and a negative value). For each $(x, y) \neq (0, 0)$, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point $(x, y)$. The other polar coordinate representations for the point can be determined from these two, as in Example 1.

**EXAMPLE 4** Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

| Polar equation | Cartesian equivalent |
|---|---|
| $r\cos\theta = 2$ | $x = 2$ |
| $r^2\cos\theta\sin\theta = 4$ | $xy = 4$ |
| $r^2\cos^2\theta - r^2\sin^2\theta = 1$ | $x^2 - y^2 = 1$ |
| $r = 1 + 2r\cos\theta$ | $y^2 - 3x^2 - 4x - 1 = 0$ |
| $r = 1 - \cos\theta$ | $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$ |

Some curves are more simply expressed with polar coordinates; others are not. ■

### Graphing Sets of Polar Coordinate Points

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 11–26.

**11.** $r = 2$

**12.** $0 \leq r \leq 2$

**13.** $r \geq 1$

**14.** $1 \leq r \leq 2$

**15.** $0 \leq \theta \leq \pi/6, \quad r \geq 0$

**16.** $\theta = 2\pi/3, \quad r \leq -2$

**17.** $\theta = \pi/3, \quad -1 \leq r \leq 3$

**18.** $\theta = 11\pi/4, \quad r \geq -1$

**19.** $\theta = \pi/2, \quad r \geq 0$

**20.** $\theta = \pi/2, \quad r \leq 0$

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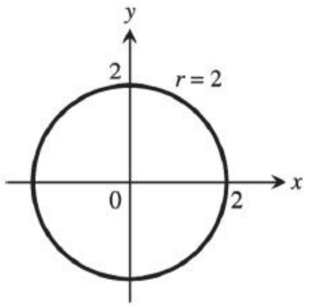
**20.** $\theta = \pi/2, \quad r \le 0$

11.

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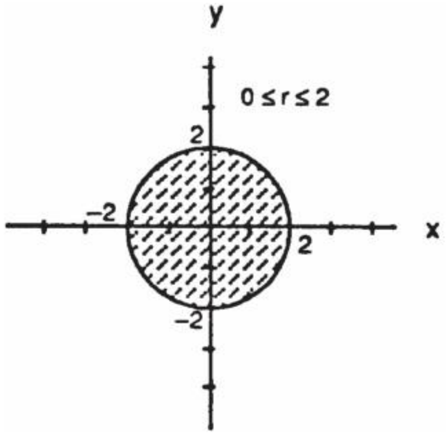


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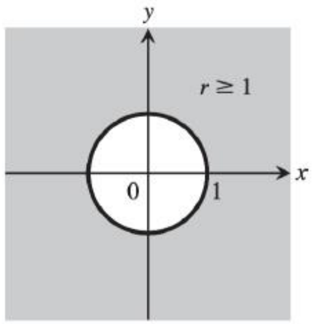


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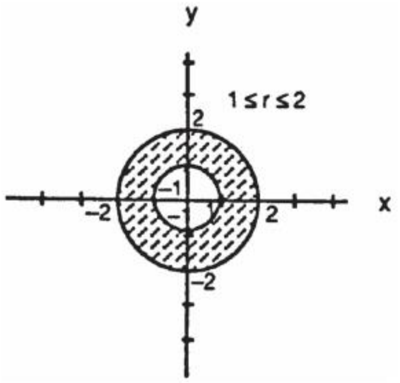


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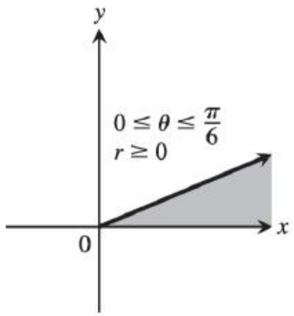


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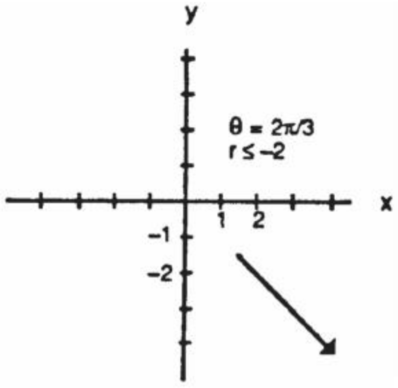

### Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

27. $r \cos \theta = 2$
28. $r \sin \theta = -1$
29. $r \sin \theta = 0$
30. $r \cos \theta = 0$
31. $r = 4 \csc \theta$
32. $r = -3 \sec \theta$
33. $r \cos \theta + r \sin \theta = 1$
34. $r \sin \theta = r \cos \theta$
35. $r^2 = 1$
36. $r^2 = 4r \sin \theta$
37. $r = \dfrac{5}{\sin \theta - 2 \cos \theta}$
38. $r^2 \sin 2\theta = 2$
39. $r = \cot \theta \csc \theta$
40. $r = 4 \tan \theta \sec \theta$
41. $r = \csc \theta \, e^{r \cos \theta}$
42. $r \sin \theta = \ln r + \ln \cos \theta$
43. $r^2 + 2r^2 \cos \theta \sin \theta = 1$
44. $\cos^2 \theta = \sin^2 \theta$
45. $r^2 = -4r \cos \theta$
46. $r^2 = -6r \sin \theta$
47. $r = 8 \sin \theta$
48. $r = 3 \cos \theta$
49. $r = 2 \cos \theta + 2 \sin \theta$
50. $r = 2 \cos \theta - \sin \theta$
51. $r \sin \left( \theta + \dfrac{\pi}{6} \right) = 2$
52. $r \sin \left( \dfrac{2\pi}{3} - \theta \right) = 5$

### Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

**27.** $r\cos\theta = 2$

**28.** $r\sin\theta = -1$

**29.** $r\sin\theta = 0$

**30.** $r\cos\theta = 0$

**31.** $r = 4\csc\theta$

**32.** $r = -3\sec\theta$

**33.** $r\cos\theta + r\sin\theta = 1$

**34.** $r\sin\theta = r\cos\theta$

**35.** $r^2 = 1$

**36.** $r^2 = 4r\sin\theta$

**37.** $r = \dfrac{5}{\sin\theta - 2\cos\theta}$

**38.** $r^2\sin 2\theta = 2$

**39.** $r = \cot\theta\csc\theta$

**40.** $r = 4\tan\theta\sec\theta$

**41.** $r = \csc\theta\, e^{r\cos\theta}$

**42.** $r\sin\theta = \ln r + \ln\cos\theta$

**43.** $r^2 + 2r^2\cos\theta\sin\theta = 1$

**44.** $\cos^2\theta = \sin^2\theta$

**45.** $r^2 = -4r\cos\theta$

**46.** $r^2 = -6r\sin\theta$

**47.** $r = 8\sin\theta$

**48.** $r = 3\cos\theta$

**49.** $r = 2\cos\theta + 2\sin\theta$

**50.** $r = 2\cos\theta - \sin\theta$

**51.** $r\sin\left(\theta + \frac{\pi}{6}\right) = 2$

**52.** $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

41. $r = (\csc\theta)e^{r\cos\theta} \Rightarrow r\sin\theta = e^{r\cos\theta} \Rightarrow y = e^x$, graph of the natural exponential function

- Questions?

We’re here to help.

Remember the tutoring center is open!

Study hard, best of luck!

Be well stay safe & healthy.