MAT112 -Mr. José Pabón

Recitation will start soon.

We will pass this course with a great grade & meet our academic and professional goals

Revolution About the *y*-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$

Revolution About the *x*-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

12. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

- MAT112 T.A. José Pabón

We will be courteous, civil to each other.
NO SUCH THING AS AN OBVIOUS QUESTION
ask ask ask any doubt to clear up

- Attendance is required.

Any absence justifications must be addressed with the office of the Dean of Students

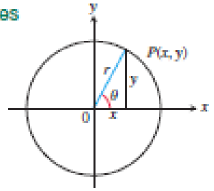
Trigonometry Formulas

Definitions and Fundamental Identities

Sine: $\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$

Cosine: $\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$

Tangent: $\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta}$



Identities

$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \frac{\pi}{2}\right) = \sin A$

$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$

$\sin A\sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$

$\cos A\cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$

$\sin A\cos B = \frac{1}{2}\sin(A - B) + \frac{1}{2}\sin(A + B)$

$\sin A + \sin B = 2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$

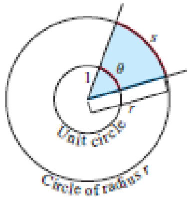
$\sin A - \sin B = 2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$

$\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$

$\cos A - \cos B = -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$

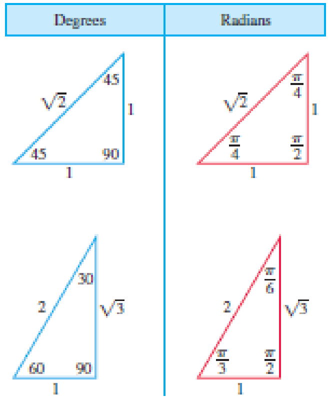
Trigonometric Functions

Radian Measure

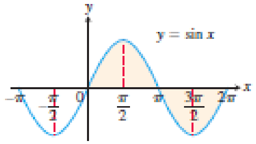


$\frac{s}{r} = \frac{\theta}{1} = \theta$ or $\theta = \frac{s}{r}$,

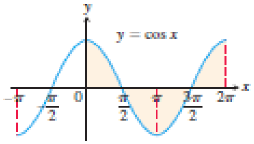
$180° = \pi$ radians.



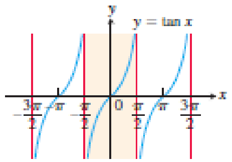
The angles of two common triangles, in degrees and radians.



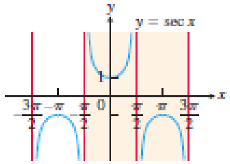
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



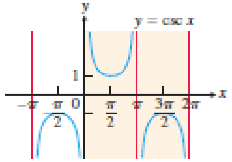
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



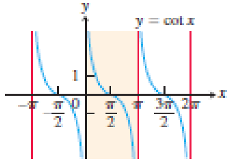
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, \infty)$

6.1 Volumes Using Cross-Sections

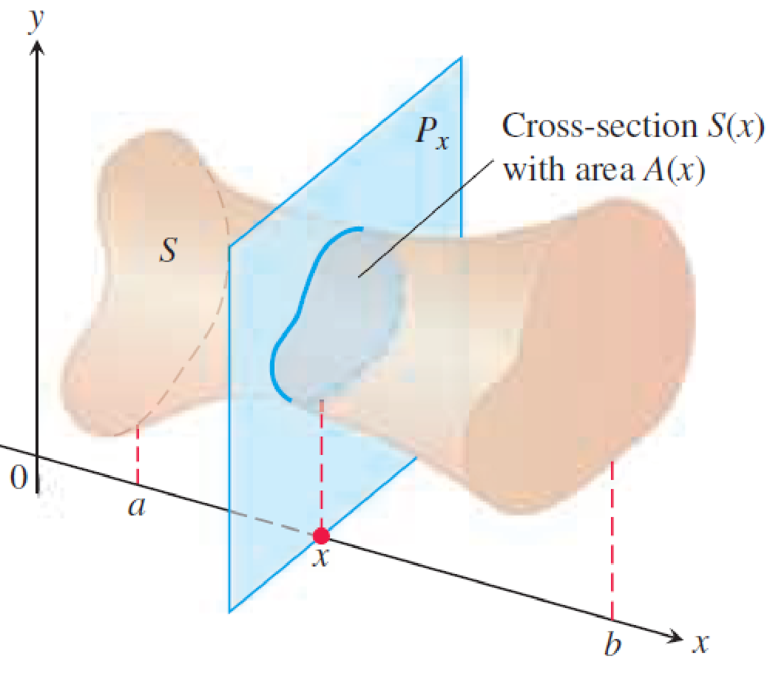
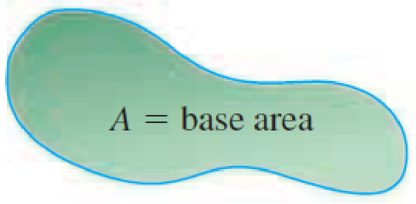


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

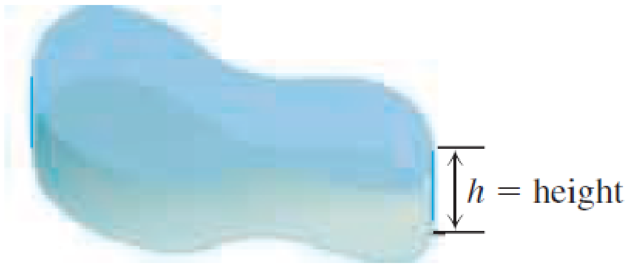
In this section we define volumes of solids by using the areas of their cross-sections. A **cross-section** of a solid S is the planar region formed by intersecting S with a plane (Figure 6.1). We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of slicing, the disk method, and the washer method.

Suppose that we want to find the volume of a solid S like the one pictured in Figure 6.1. At each point x in the interval $[a, b]$ we form a cross-section $S(x)$ by intersecting S with a plane perpendicular to the x-axis through the point x, which gives a planar region whose area is $A(x)$. We will show that if A is a continuous function of x, then the volume of the solid S is the definite integral of $A(x)$. This method of computing volumes is known as the **method of slicing**.

Before showing how this method works, we need to extend the definition of a cylinder from the usual cylinders of classical geometry (which have circular, square, or other regular bases) to cylindrical solids that have more general bases. As shown in Figure 6.2, if the



Plane region whose area we know



Cylindrical solid based on region
Volume = base area × height = Ah

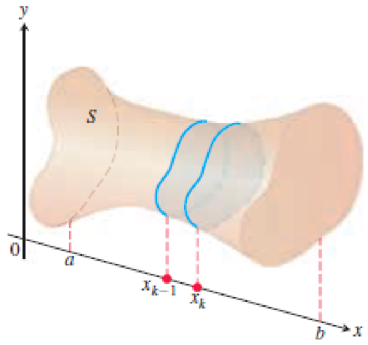


FIGURE 6.3 A typical thin slab in the solid S.

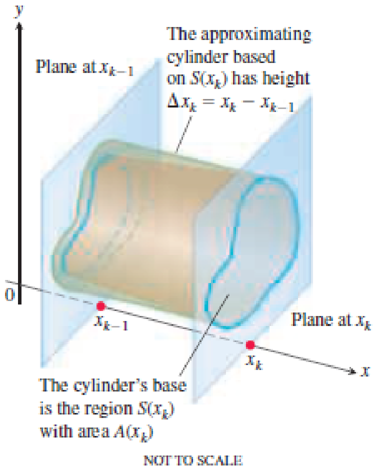


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

cylindrical solid has a base whose area is A and its height is h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

In the method of slicing, the base will be the cross-section of S that has area $A(x)$, and the height will correspond to the width Δx_k of subintervals formed by partitioning the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. These planes slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. The approximation given by this Riemann sum converges to the definite integral of $A(x)$ as $n \to \infty$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$

Therefore, we define this definite integral to be the volume of the solid S.

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

This definition applies whenever $A(x)$ is integrable, and in particular when $A(x)$ is continuous. To apply this definition to calculate the volume of a solid using cross-sections perpendicular to the x-axis, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, *the area of a typical cross-section.*
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ *to find the volume.*

Solids of Revolution: The Disk Method

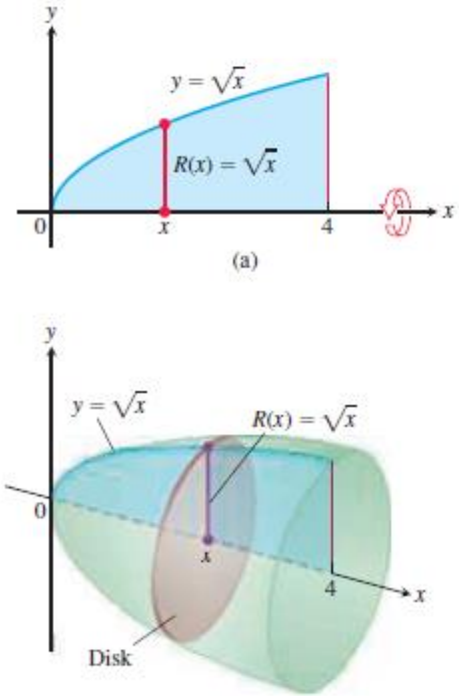
The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we first observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, where $R(x)$ is the distance from the axis of revolution to the planar region's boundary. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

Therefore, the definition of volume gives us the following formula.

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$



This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx \\ &= \int_0^4 \pi[\sqrt{x}]^2\,dx \qquad \text{Radius } R(x) = \sqrt{x} \text{ for rotation around } x\text{-axis.} \\ &= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \end{aligned}$$

■

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \leq y \leq d$, about the y-axis, we use the same method with x replaced by y. In this case, the area of the circular cross-section is

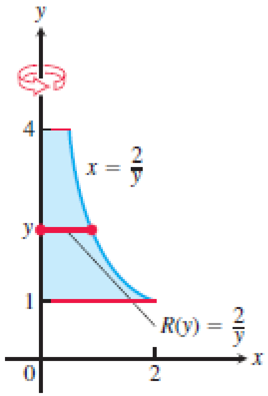
$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2,$$

and the definition of volume gives us the following formula.

Volume by Disks for Rotation About the y-Axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y-axis.



Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(y)]^2\,dy \\ &= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \qquad \text{Radius } R(y) = \frac{2}{y} \text{ for rotation around } y\text{-axis} \\ &= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] = 3\pi. \end{aligned}$$

■

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, then the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius: } R(x)$$
$$\text{Inner radius: } r(x)$$

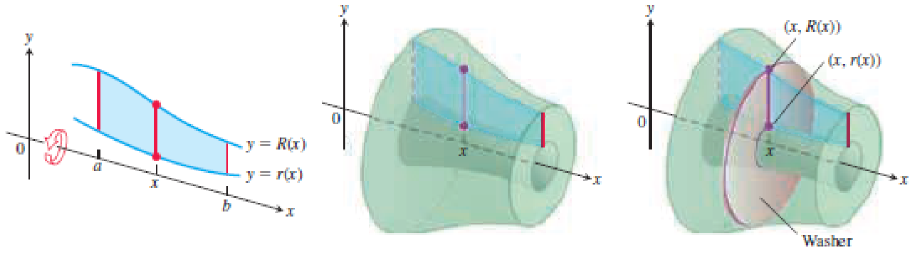


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

The washer's area is the area of a circle of radius $R(x)$ minus the area of a circle of radius $r(x)$:

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

Consequently, the definition of volume in this case gives us the following formula.

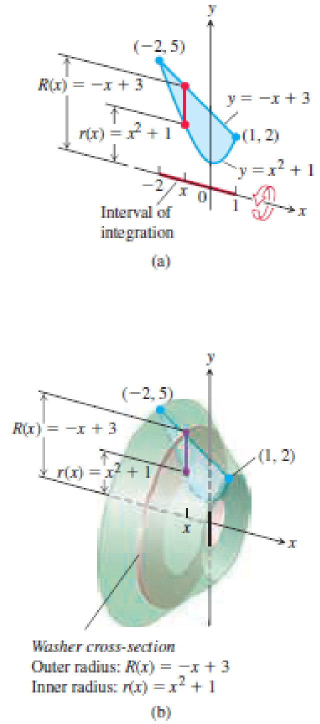


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right)dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a thin slab of the solid resembles a circular washer with outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution We use the same four steps for calculating the volume of a solid that were discussed earlier in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14a).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

   These radii are the distances of the ends of the line segment from the axis of revolution (see Figure 6.14).

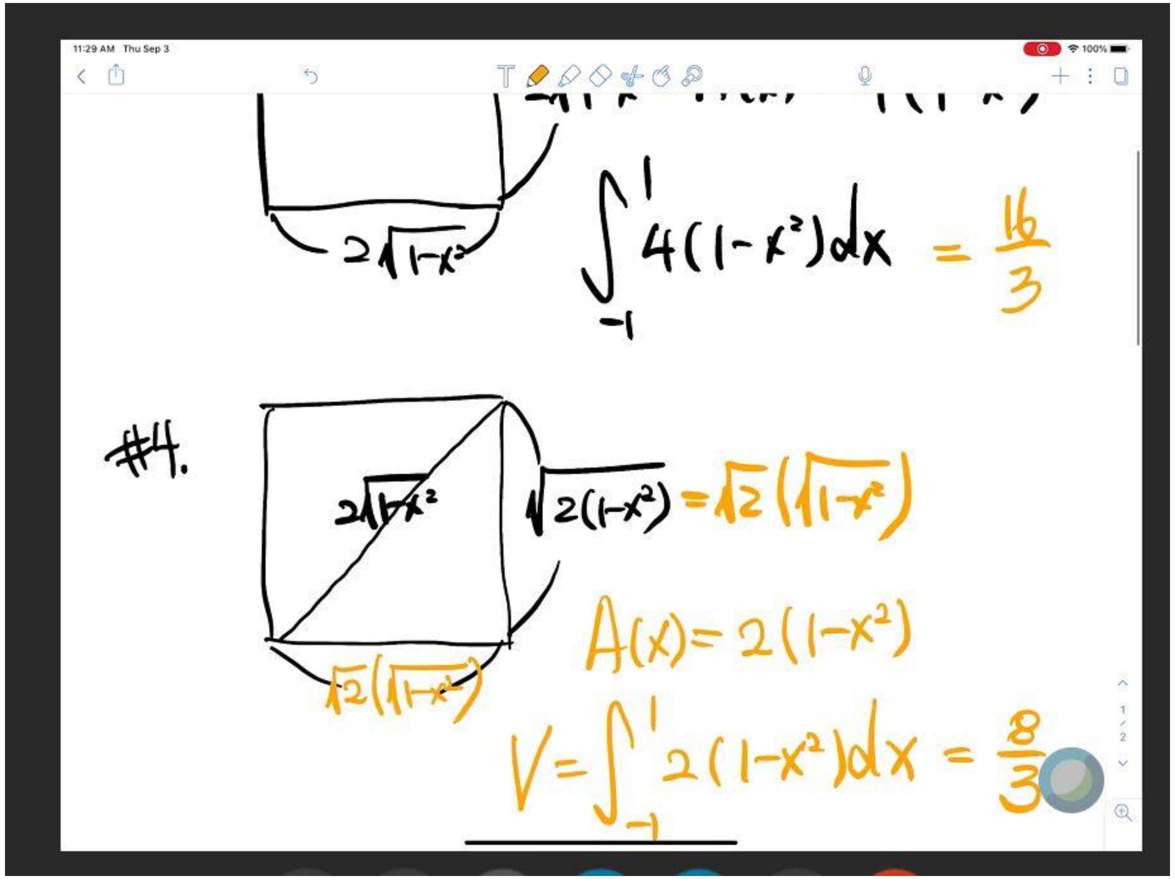
$$\text{Outer radius:}\quad R(x) = -x + 3$$
$$\text{Inner radius:}\quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$x^2 + 1 = -x + 3$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, \quad x = 1 \qquad \text{Limits of integration}$$

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.



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4. $A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left[\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right]^2}{2} = \frac{\left(2\sqrt{1-x^2}\right)^2}{2} = 2\left(1-x^2\right); \quad a = -1, b = 1;$

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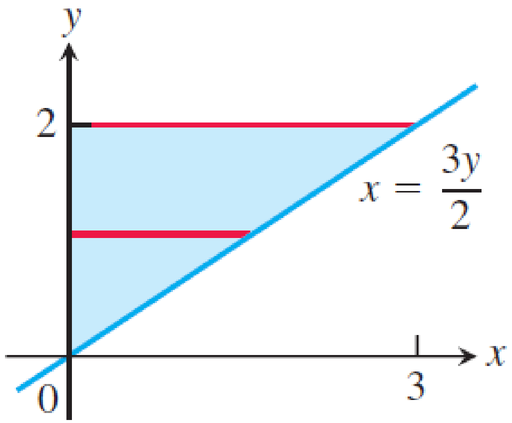
$$V = \int_a^b A(x)\,dx = 2\int_{-1}^{1}\left(1-x^2\right)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 4\left(1-\tfrac{1}{3}\right) = \tfrac{8}{3}$$

Volumes by the Disk Method

In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

18. About the y-axis



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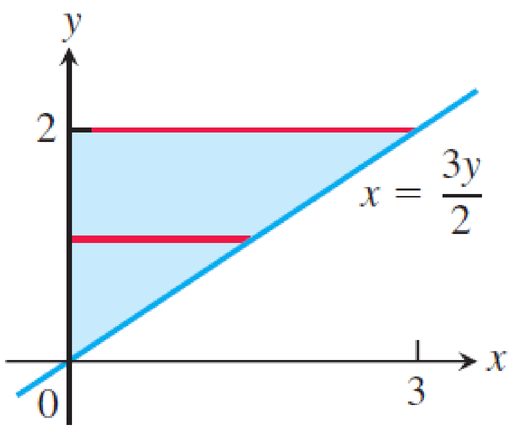

18. $R(y) = x = \frac{3y}{2} \Rightarrow$

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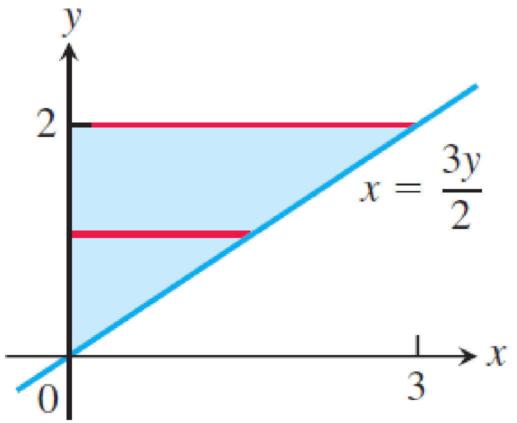

18. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy =$

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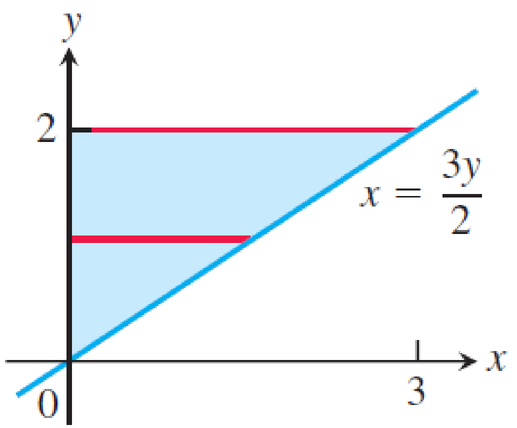
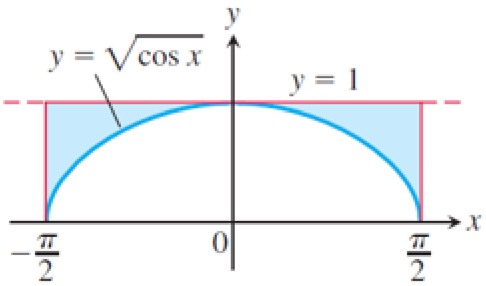


18. $R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi \int_0^2 \frac{9}{4} y^2 dy = \pi \left[\frac{3}{4} y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

39. The x-axis

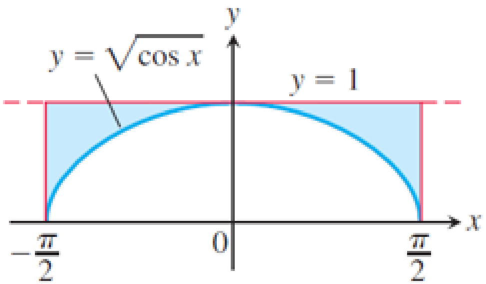


40.

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

39. The x-axis **40.**



39. For the sketch given, $a = -\frac{\pi}{2}, b = \frac{\pi}{2}; R(x) = 1, r(x) = \sqrt{\cos x};$

Volumes by the Washer Method

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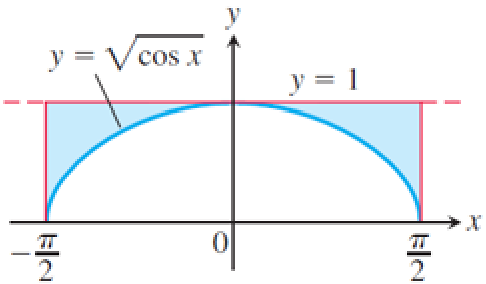


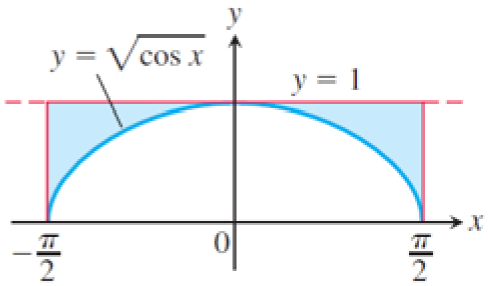
39. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

39. The x-axis

40.



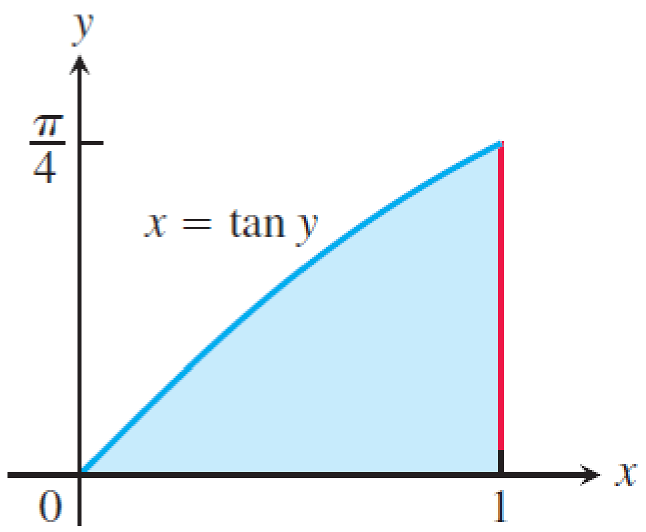
39. For the sketch given, $a = -\frac{\pi}{2}$, $b = \frac{\pi}{2}$; $R(x) = 1$, $r(x) = \sqrt{\cos x}$; $V = \int_a^b \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$$= \int_{-\pi/2}^{\pi/2} \pi(1 - \cos x)\, dx = 2\pi \int_0^{\pi/2} (1 - \cos x)\, dx = 2\pi\left[x - \sin x\right]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi$$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

40. The y-axis



Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

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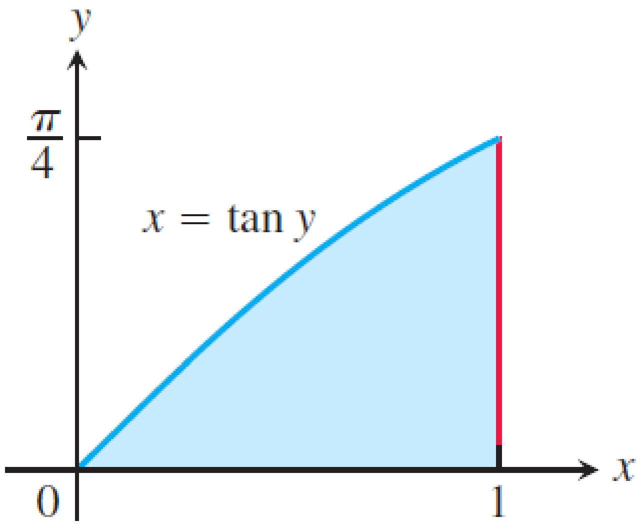
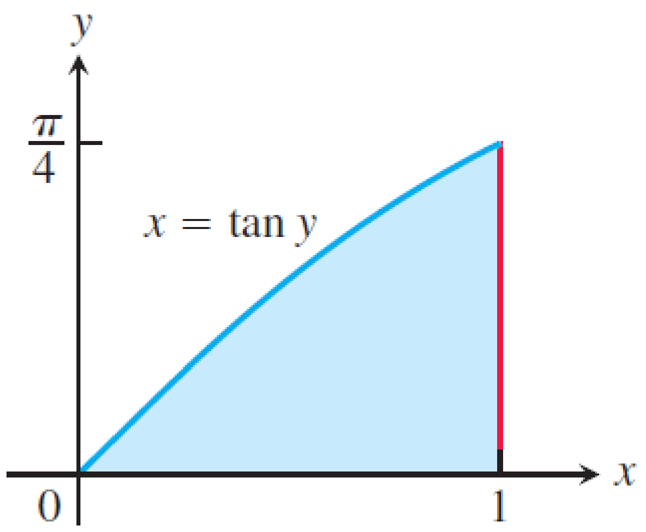


40. For the sketch given, $c = 0,\ d = \frac{\pi}{4};\ R(y) = 1,\ r(y) = \tan y;\ V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

40. The y-axis



40. For the sketch given, $c = 0, d = \frac{\pi}{4}$; $R(y) = 1, r(y) = \tan y$; $V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$$= \pi\int_0^{\pi/4}\left(1 - \tan^2 y\right) dy = \pi\int_0^{\pi/4}\left(2 - \sec^2 y\right) dy = \pi\left[2y - \tan y\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$$

6.2 Volumes Using Cylindrical Shells

In Section 6.1 we defined the volume of a solid to be the definite integral $V = \int_a^b A(x)\,dx$, where $A(x)$ is an integrable cross-sectional area of the solid from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. However, this method of slicing is sometimes awkward to apply, as we will illustrate in our first example. To overcome this difficulty, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.

Slicing with Cylinders

Suppose we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid so that the axis of each cylinder is parallel to the y-axis. The vertical axis of each cylinder is always the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This slab interpretation allows us to apply the same integral definition for volume as before. The following example provides some insight.

$$\Delta V_k = \text{circumference} \times \text{height} \times \text{thickness}$$

The Shell Method

Suppose that the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (see Figure 6.19a). We assume $a \geq L$, so the vertical line may touch the region but cannot pass through it. We generate a solid S by rotating this region about the vertical line L.

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$. As usual, we choose a point c_k in each subinterval $[x_{k-1}, x_k]$. In Example 1 we chose c_k to be the endpoint x_k, but now it will be more convenient to let c_k be the midpoint of the subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.19a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width

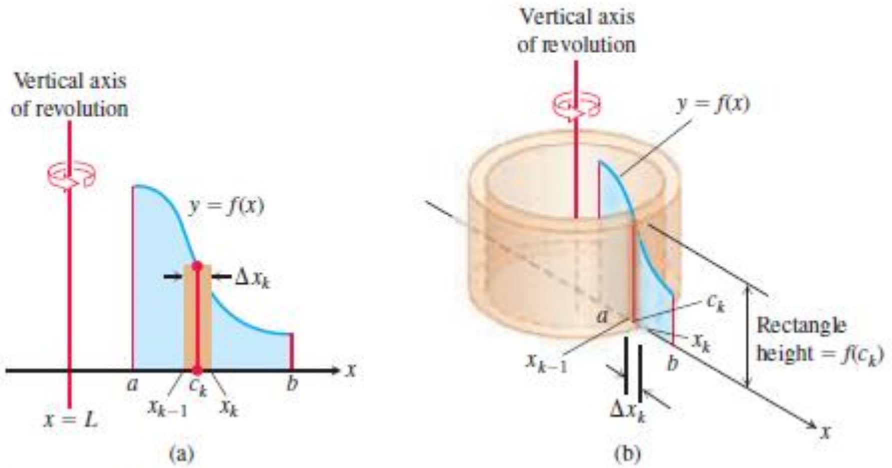


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

The volume of a cylindrical shell of height h with inner radius r and outer radius R is

$$\pi R^2 h - \pi r^2 h = 2\pi\left(\frac{R+r}{2}\right)(h)(R-r).$$

$\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.19b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$\begin{aligned}\Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness}\\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \qquad R = x_k - L \text{ and } r = x_{k-1} - L\end{aligned}$$

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles:

$$V \approx \sum_{k=1}^{n} \Delta V_k.$$

The limit of this Riemann sum as each $\Delta x_k \to 0$ and $n \to \infty$ gives the volume of the solid as a definite integral:

$$\begin{aligned}V = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k &= \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx\\ &= \int_a^b 2\pi(x - L)f(x)\, dx.\end{aligned}$$

We refer to the variable of integration, here x, as the **thickness variable**. To emphasize the *process* of the shell method, we state the general formula in terms of the shell radius and shell height. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx.$$

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

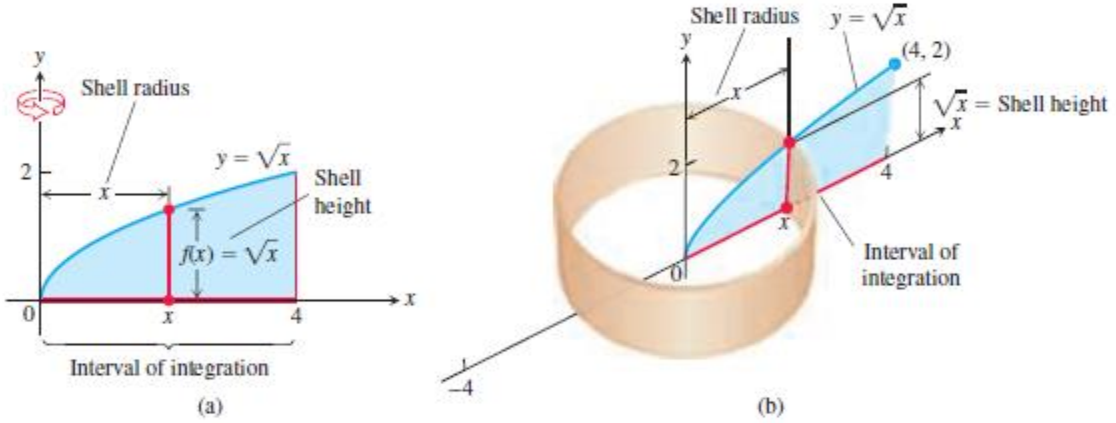


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx$$

$$= \int_0^4 2\pi(x)(\sqrt{x})\, dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 4 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.21). The volume of the solid is

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$

$$= \int_0^2 2\pi(y)(4 - y^2)\, dy$$

$$= 2\pi \int_0^2 (4y - y^3)\, dy$$

$$= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi.$$

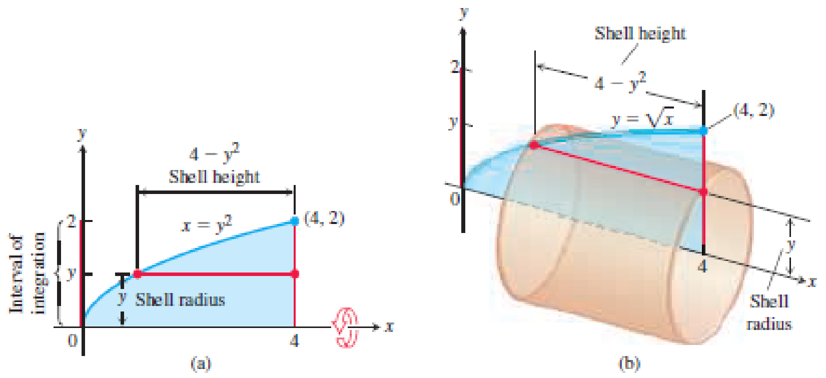


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy. ■

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. (Exercise 45 outlines a proof.) Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

12. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

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Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

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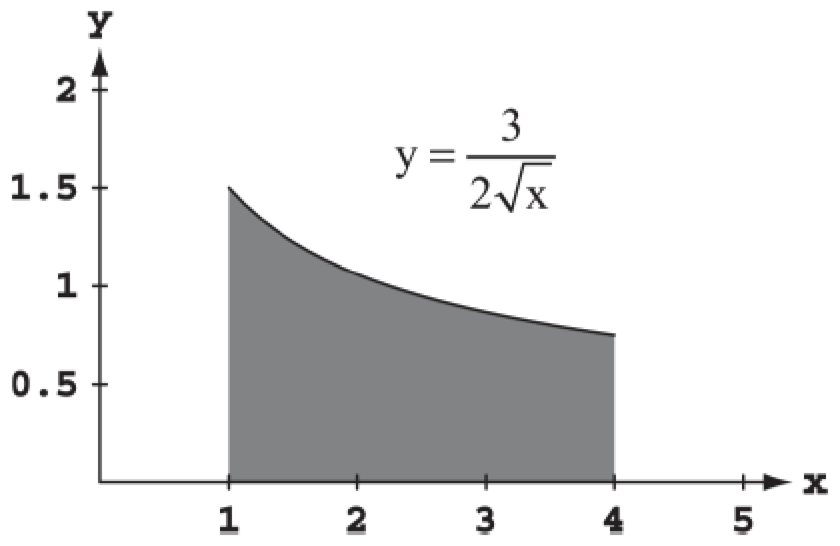

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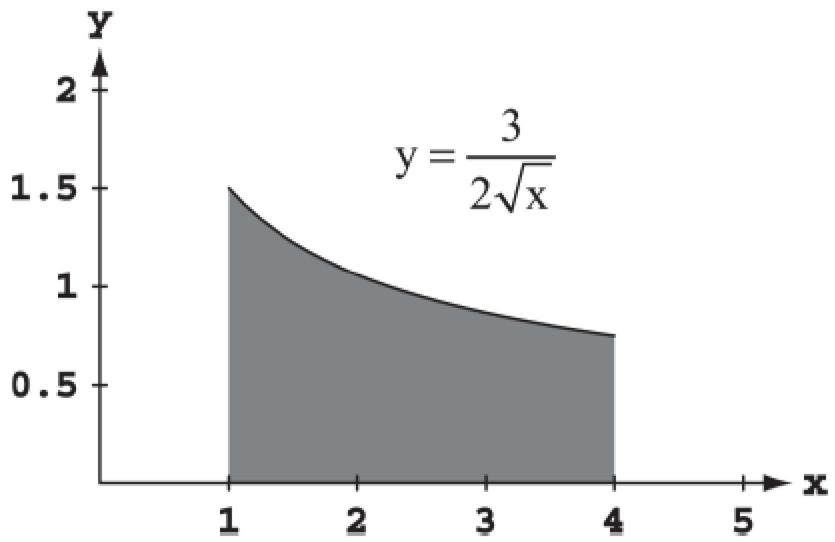

12. $a = 1, b = 4;$

$$V = \int_a^b 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx = \int_1^4 2\pi x \left(\frac{3}{2}\, x^{-1/2}\right) dx$$

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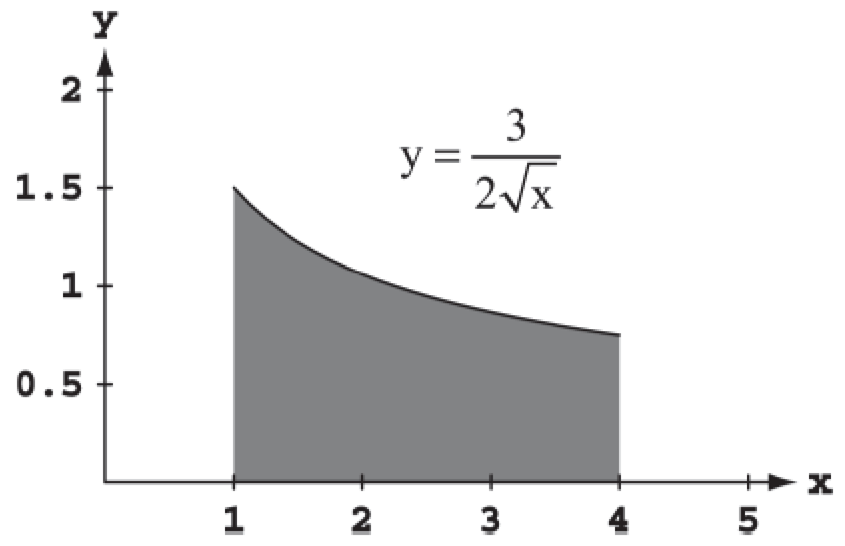

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$$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi \left[\tfrac{2}{3}\, x^{3/2}\right]_1^4 = 2\pi \left(4^{3/2} - 1\right)$$

Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

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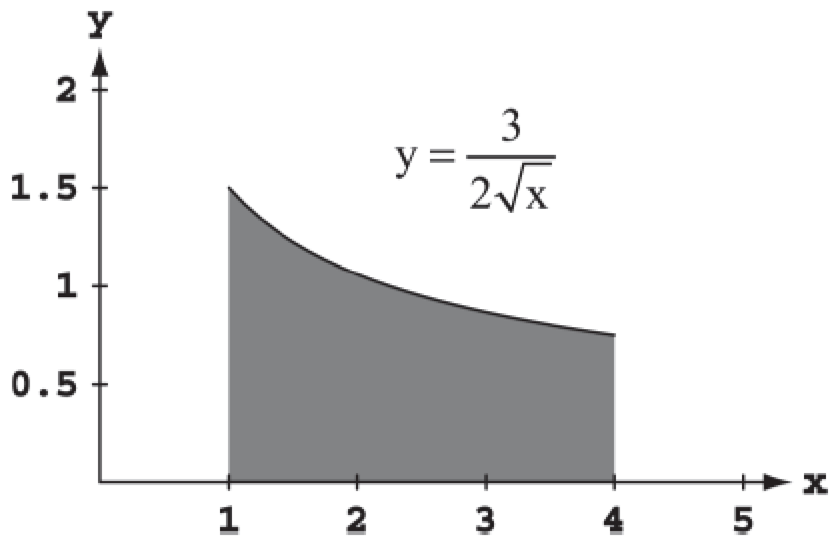

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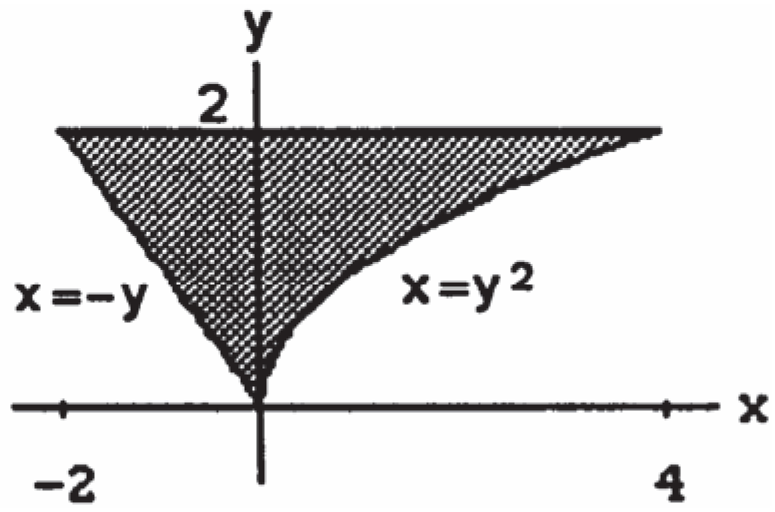
$$= 3\pi \int_1^4 x^{1/2}\, dx = 3\pi \left[\tfrac{2}{3}\, x^{3/2}\right]_1^4 = 2\pi\left(4^{3/2} - 1\right)$$

$$= 2\pi(8-1) = 14\pi$$

Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$



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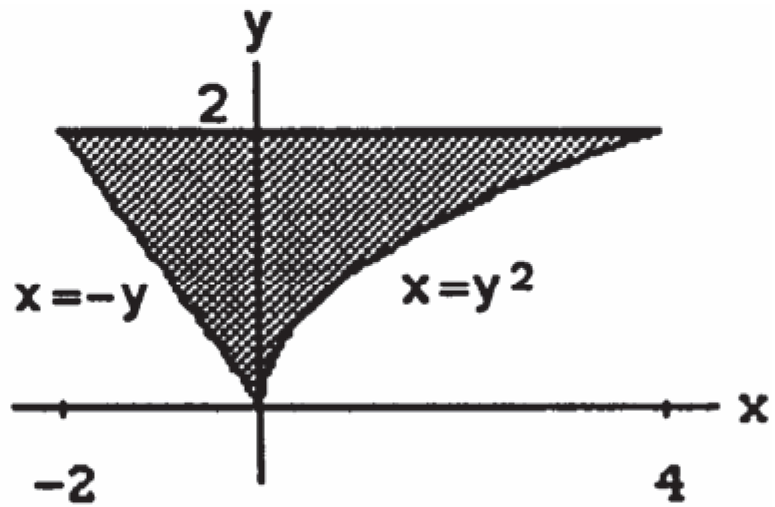

16. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy = \int_0^2 2\pi y \left[y^2 - (-y) \right] dy$$

Revolution About the x-Axis

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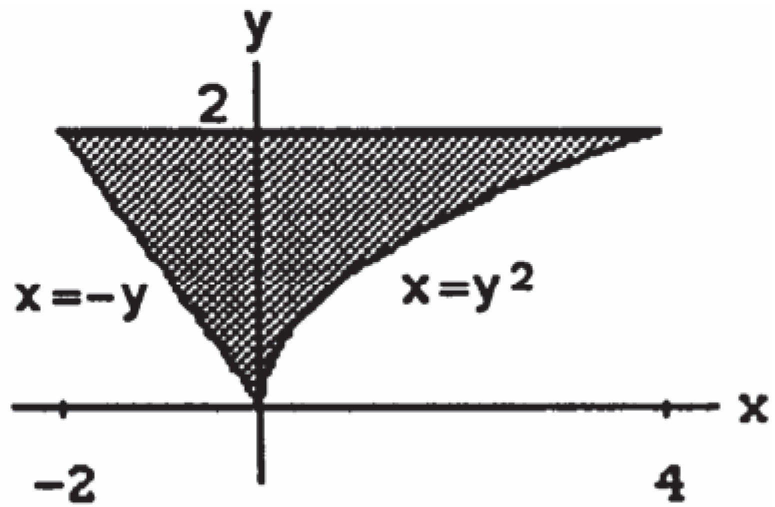

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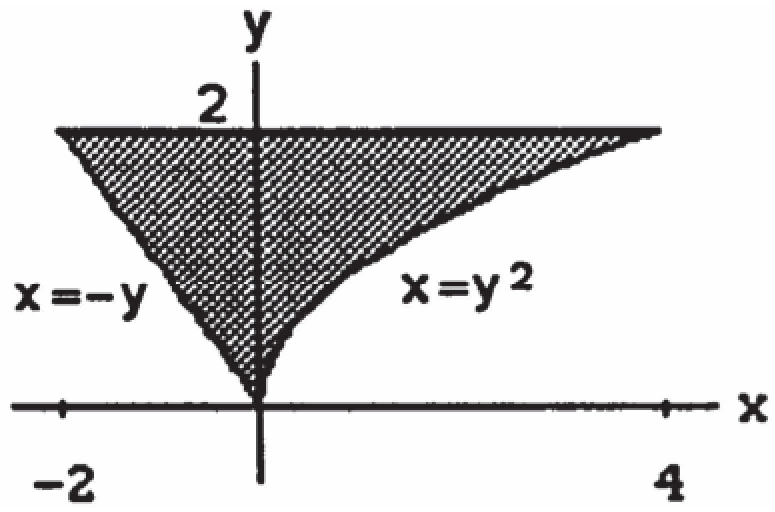
$$= 2\pi \int_0^2 \left(y^3 + y^2 \right) dy = 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3} \right]_0^2 = 16\pi \left(\tfrac{2}{4} + \tfrac{1}{3} \right)$$

Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.



16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \geq 0$

16. $c = 0, d = 2;$

$$V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dy = \int_0^2 2\pi y \left[y^2 - (-y) \right] dy$$

$$= 2\pi \int_0^2 \left(y^3 + y^2 \right) dy = 2\pi \left[\frac{y^4}{4} + \frac{y^3}{3} \right]_0^2 = 16\pi \left(\frac{2}{4} + \frac{1}{3} \right)$$

$$= 16\pi \left(\frac{5}{6} \right) = \frac{40\pi}{3}$$

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

EXAMPLE 1 Find the length of the curve shown in Figure 6.24, which is the graph of the function

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (3) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad \text{If } x = 1, \text{ then } y \approx 0.89$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (3) with $a = 0, b = 1$.

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17.$$

Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$.