MAT112 -Mr. José Pabón

Recitation will start soon.

We will pass this course with a great grade & meet our academic and professional goals

3. $\int (\sec x - \tan x)^2 \, dx$

8. $\int \frac{2^{\ln z^3}}{16z} \, dz$

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2} \, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

4. $\int x^2 \sin x \, dx$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int xe^x \, dx$

8. $\int xe^{3x} \, dx$

- MAT112 T.A. José Pabón

We will be courteous, civil to each other.
NO SUCH THING AS AN OBVIOUS QUESTION
ask ask ask any doubt to clear up

Integrity – Personal, Academic, Professsional.

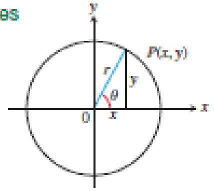
Trigonometry Formulas

Definitions and Fundamental Identities

Sine: $\sin\theta = \dfrac{y}{r} = \dfrac{1}{\csc\theta}$

Cosine: $\cos\theta = \dfrac{x}{r} = \dfrac{1}{\sec\theta}$

Tangent: $\tan\theta = \dfrac{y}{x} = \dfrac{1}{\cot\theta}$



Identities

$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$

$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$\sin\left(A - \dfrac{\pi}{2}\right) = -\cos A, \quad \cos\left(A - \dfrac{\pi}{2}\right) = \sin A$

$\sin\left(A + \dfrac{\pi}{2}\right) = \cos A, \quad \cos\left(A + \dfrac{\pi}{2}\right) = -\sin A$

$\sin A\sin B = \dfrac{1}{2}\cos(A - B) - \dfrac{1}{2}\cos(A + B)$

$\cos A\cos B = \dfrac{1}{2}\cos(A - B) + \dfrac{1}{2}\cos(A + B)$

$\sin A\cos B = \dfrac{1}{2}\sin(A - B) + \dfrac{1}{2}\sin(A + B)$

$\sin A + \sin B = 2\sin\dfrac{1}{2}(A + B)\cos\dfrac{1}{2}(A - B)$

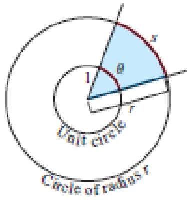
$\sin A - \sin B = 2\cos\dfrac{1}{2}(A + B)\sin\dfrac{1}{2}(A - B)$

$\cos A + \cos B = 2\cos\dfrac{1}{2}(A + B)\cos\dfrac{1}{2}(A - B)$

$\cos A - \cos B = -2\sin\dfrac{1}{2}(A + B)\sin\dfrac{1}{2}(A - B)$

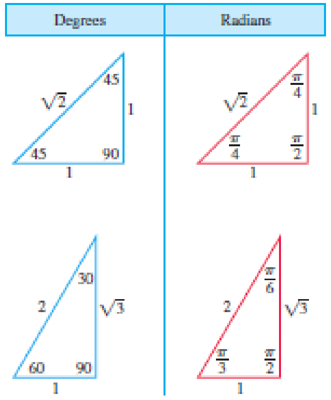
Trigonometric Functions

Radian Measure

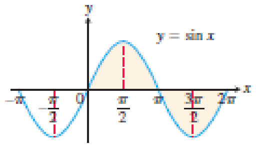


$\dfrac{s}{r} = \dfrac{\theta}{1} = \theta$ or $\theta = \dfrac{s}{r}$,

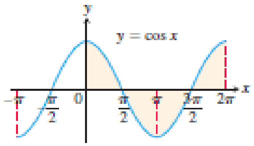
$180° = \pi$ radians.



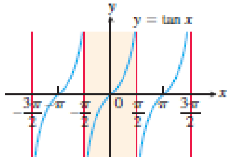
The angles of two common triangles, in degrees and radians.



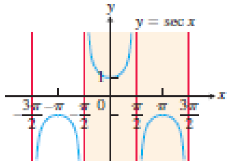
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



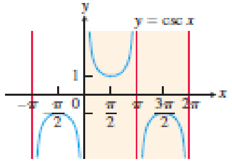
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



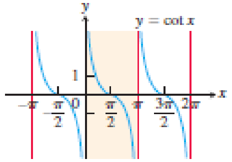
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, \infty)$

In[13]:= m

Out[13]= $\frac{1}{\sqrt{y}} + \sqrt{y}$

In[14]:= m^2

Out[14]= $\left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)^2$

In[15]:= Simplify[m^2]

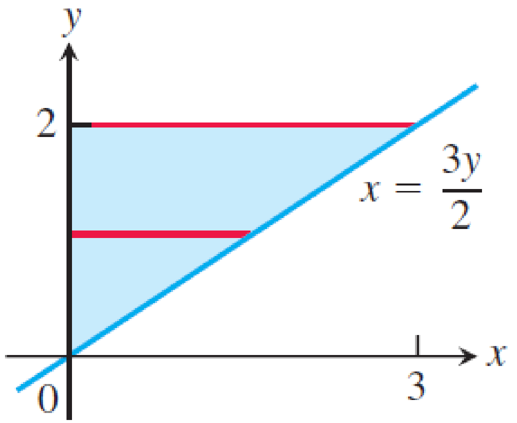
Out[15]= $2 + \frac{1}{y} + y$

$$2 + \frac{1}{y} + y = \left(\frac{1}{\sqrt{y}} + \sqrt{y}\right)^2$$

Volumes by the Disk Method

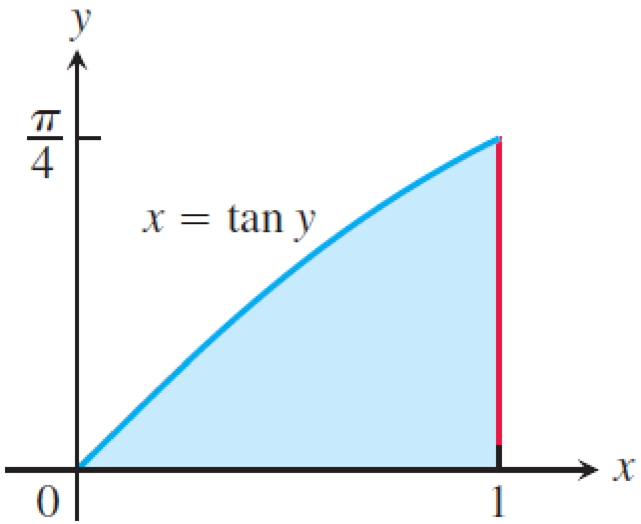
In Exercises 17–20, find the volume of the solid generated by revolving the shaded region about the given axis.

18. About the y-axis



18. $$R(y) = x = \frac{3y}{2} \Rightarrow V = \int_0^2 \pi [R(y)]^2 dy = \pi \int_0^2 \left(\frac{3y}{2}\right)^2 dy = \pi \int_0^2 \frac{9}{4} y^2 dy = \pi \left[\frac{3}{4} y^3\right]_0^2 = \pi \cdot \frac{3}{4} \cdot 8 = 6\pi$$

40. The y-axis



Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 39 and 40 about the indicated axes.

40. For the sketch given, $c = 0,\ d = \frac{\pi}{4};\ R(y) = 1,\ r(y) = \tan y;\ V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$$= \pi\int_0^{\pi/4}\left(1 - \tan^2 y\right) dy = \pi\int_0^{\pi/4}\left(2 - \sec^2 y\right) dy = \pi\left[2y - \tan y\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi^2}{2} - \pi$$

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

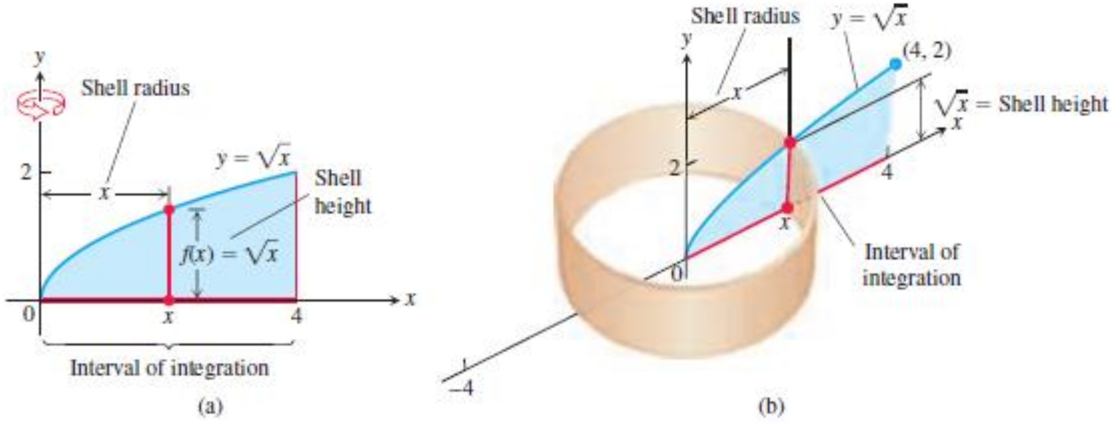


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is

$$V = \int_a^b 2\pi \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right) dx$$

$$= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}.$$

■

Note that the square root in Equation (3) is similar to the one that appears in the formula for the arc length differential of the generating curve in Equation (6) of Section 6.3.

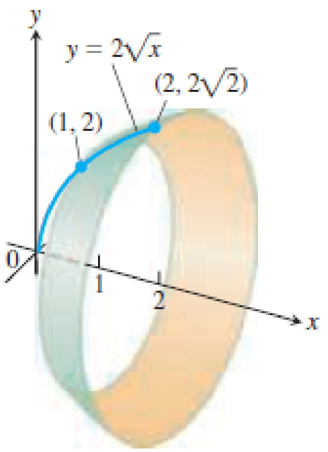


FIGURE 6.34 In Example 1 we calculate the area of this surface.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure 6.34).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

With these substitutions, we have

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$ ■

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{2}$$

Hooke's Law for Springs: $F = kx$

One calculation for work arises in finding the work required to stretch or compress a spring. **Hooke's Law** says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{3}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

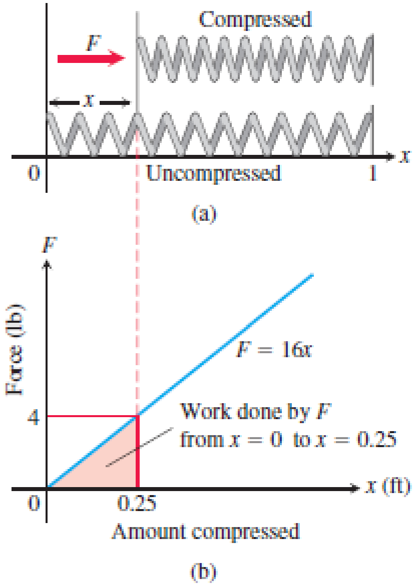


FIGURE 6.36 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

EXAMPLE 2 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.36). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \quad \text{to} \quad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft-lb}.$$

Eq. (2) with $a = 0, b = 0.25, F(x) = 16x$ ■

7.3 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications.

Definitions and Identities

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $\sinh x$ as "cinch x," rhyming with "pinch x," and $\cosh x$ as "kosh x," rhyming with "gosh x." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 7.4. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

Hyperbolic functions satisfy the identities in Table 7.5. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} && \text{Simplify.} \\ &= \sinh 2x. && \text{Definition of sinh} \end{aligned}$$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra.

For any real number u, we know the point with coordinates $(\cos u, \sin u)$ lies on the unit circle $x^2 + y^2 = 1$. So the trigonometric functions are sometimes called the *circular* functions. Because of the first identity

$$\cosh^2 u - \sinh^2 u = 1,$$

with u substituted for x in Table 7.5, the point having coordinates $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where the *hyperbolic* functions get their names (see Exercise 86).

TABLE 7.5 Identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

TABLE 7.6 Derivatives of hyperbolic functions

$\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

$\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

$\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

$\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

$\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

$\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

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Hyperbolic functions are useful in finding integrals, which we will see in Chapter 8. They play an important role in science and engineering as well. The hyperbolic cosine describes the shape of a hanging cable or wire that is strung between two points at the same height and hanging freely (see Exercise 83). The shape of the St. Louis Arch is an inverted hyperbolic cosine. The hyperbolic tangent occurs in the formula for the velocity of an ocean wave moving over water having a constant depth, and the inverse hyperbolic tangent describes how relative velocities sum according to Einstein's Law in the Special Theory of Relativity.

Derivatives and Integrals of Hyperbolic Functions

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.6). Again, there are similarities with trigonometric functions.

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned} \frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\ &= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\ &= \cosh u \frac{du}{dx}. && \text{Definition of } \cosh u \end{aligned}$$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.10}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3 + 4x^2}} = \int \frac{du}{\sqrt{a^2 + u^2}} \qquad u = 2x,\quad du = 2\,dx,\quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.10}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

■

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$. This procedure transforms the integral into one we can evaluate:

$$\int_0^{\pi/4} \frac{dx}{1 - \sin x} = \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx$$

Multiply and divide by conjugate.

$$= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx$$

Simplify.

$$= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx$$

$1 - \sin^2 x = \cos^2 x$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx$$

Use Table 8.1, Formulas 8 and 10

$$= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \left(1 + \sqrt{2} - (0 + 1)\right) = \sqrt{2}.$$

EXAMPLE 6 Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then we obtain

$$3\int \frac{x\,dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du$$

$$= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1.$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1} x + C_2. \qquad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1} x + C.$$ ■

The integral of any odd function over any symmetric interval is always zero.

$$\int_{-1}^{1} x^3 dx = \frac{1}{4}x^4|_{-1}^{1} = \frac{1}{4}(1^4 - (-1^4)) = \frac{1}{4}((1) - (1)) = \frac{1}{4} \cdot 0 = 0 \qquad (1)$$

Same for:

$$\int_{-5634}^{5634} x^3 dx = \frac{1}{4}x^4|_{-5634}^{5634} = \frac{1}{4}(5634^4 - (-5634^4)) = \frac{1}{4}(5634^4 - (5634^4)) = \frac{1}{4}(0) == \frac{1}{4} \cdot 0 = 0 \qquad (2)$$

As well as

$$\begin{aligned}\int_{-5634}^{5634} x^{333333} dx &= \frac{1}{333334}x^{333334}|_{-5634}^{5634} \\ &= \frac{1}{333334}(5634^{333334} - (-5634^{333334})) \\ &= \frac{1}{333334}(5634^{333334} - (5634^{333334})) = \frac{1}{333334}(0) = \frac{1}{333334} \cdot 0 = 0 \end{aligned} \qquad (3)$$

3. $\int (\sec x - \tan x)^2 \, dx$

3. $\int (\sec x - \tan x)^2 \, dx$

Expand the integrand: $(\sec x - \tan x)^2 = \sec^2 x - 2\sec x \tan x + \tan^2 x$

$$= \sec^2 x - 2\sec x \tan x + (\sec^2 x - 1)$$

$$= 2\sec^2 x - 2\sec x \tan x - 1$$

3. $\int (\sec x - \tan x)^2 \, dx$

Expand the integrand: $(\sec x - \tan x)^2 = \sec^2 x - 2\sec x \tan x + \tan^2 x$

$$= \sec^2 x - 2\sec x \tan x + (\sec^2 x - 1)$$

$$= 2\sec^2 x - 2\sec x \tan x - 1$$

$$\int (\sec x - \tan x)^2 \, dx = 2\int \sec^2 x \, dx - 2\int \sec x \tan x \, dx - \int 1 \, dx$$

$$= 2\tan x - 2\sec x - x + C$$

We have used Formulas 8 and 10 from Table 8.1.

8. $\int \frac{2^{\ln z^3}}{16z}\,dz$

8. $$\int \frac{2^{\ln z^3}}{16z}\,dz$$

$$u = \ln z^3 = 3\ln z \qquad du = \frac{3}{z}\,dz$$

Using Formula 5 in Table 8.1,

$$\int \frac{2^{\ln z^3}}{16z}\,dz = \frac{1}{48}\int 2^u\,du$$

8. $\displaystyle\int \frac{2^{\ln z^3}}{16z}\, dz$

$$u = \ln z^3 = 3\ln z \qquad du = \frac{3}{z}\, dz$$

Using Formula 5 in Table 8.1,

$$\int \frac{2^{\ln z^3}}{16z} dz = \frac{1}{48}\int 2^u\, du$$

$$= \frac{2^u}{48\ln 2} + C = \frac{2^{\ln z^3}}{48\ln 2} + C$$

Integration by Parts Formula

$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx \qquad (1)$$

This formula allows us to exchange the problem of computing the integral $\int u(x)\,v'(x)\,dx$ with the problem of computing a different integral, $\int v(x)\,u'(x)\,dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v, and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x)\,dx = dv$ and $u'(x)\,dx = du$, the integration by parts formula becomes

Integration by Parts Formula—Differential Version

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution There is no obvious antiderivative of $x \cos x$, so we use the integration by parts formula

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx$$

to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. In this case we factor the expression $x \cos x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = \sin x.$$

When finding an antiderivative for $v'(x)$ we have a choice of how to pick a constant of integration C. We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\int \underbrace{x}_{u(x)} \underbrace{\cos x\, dx}_{v'(x)} = \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)} \underbrace{(1)}_{u'(x)}\, dx \qquad \text{Integration by parts formula}$$

$$= x \sin x + \cos x + C \qquad \text{Integrate and simplify.}$$

and we have found the integral of the original function.

There are four apparent choices available for $u(x)$ and $v'(x)$ in Example 1:

1. Let $u(x) = 1$ and $v'(x) = x \cos x$.
2. Let $u(x) = x$ and $v'(x) = \cos x$.
3. Let $u(x) = x \cos x$ and $v'(x) = 1$.
4. Let $u(x) = \cos x$ and $v'(x) = x$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $u'(x) = \cos x - x \sin x$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\, dx.$$

The goal of integration by parts is to go from an integral $\int u(x)v'(x)\, dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x)\, dx$ that we can evaluate. Generally, you choose $v'(x)$ first to be as much of the integrand as we can readily integrate; $u(x)$ is the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one; no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

4. $\int x^2 \sin x \, dx$

5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int xe^x \, dx$

8. $\int xe^{3x} \, dx$

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

2. $u = \theta,\ du = d\theta;\ dv = \cos \pi\theta \, d\theta,\ v = \frac{1}{\pi} \sin \pi\theta;$

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

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2. $\int \theta \cos \pi\theta \, d\theta$

2. $u = \theta,\ du = d\theta;\ dv = \cos \pi\theta \, d\theta,\ v = \frac{1}{\pi} \sin \pi\theta;$

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta - \int \frac{1}{\pi} \sin \pi\theta \, d\theta = \frac{\theta}{\pi} \sin \pi\theta + \frac{1}{\pi^2} \cos \pi\theta + C$$

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Evaluate the integrals in Exercises 1–24 using integration by parts.

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6. $\int_1^e x^3 \ln x \, dx$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\ dx,\ v = \frac{x^4}{4};$

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4. $\int x^2 \sin x\, dx$

5. $\int_1^2 x \ln x\, dx$

6. $\int_1^e x^3 \ln x\, dx$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\, dx,\ v = \frac{x^4}{4};$

$$\int_1^e x^3 \ln x\, dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

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6. $\int_1^e x^3 \ln x\, dx$

7. $\int xe^x\, dx$

8. $\int xe^{3x}\, dx$

8. $u = x,\ du = dx;\ dv = e^{3x}\, dx,\ v = \frac{1}{3}e^{3x};$

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$

2. $\int \theta \cos \pi\theta \, d\theta$

3. $\int t^2 \cos t \, dt$

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5. $\int_1^2 x \ln x \, dx$

6. $\int_1^e x^3 \ln x \, dx$

7. $\int xe^x \, dx$

8. $\int xe^{3x} \, dx$

8. $u = x,\ du = dx;\ dv = e^{3x}dx,\ v = \frac{1}{3}e^{3x};$

$$\int x\, e^{3x} dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x} dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$

- Questions?

We’re here to help.
Remember the tutoring center is open!
Study hard, best of luck!