Engineering Academy

Dr. José L. Pabón, PhD

Class will start soon.

We will pass this course with a great grade & meet our academic and professional goals!

Current students

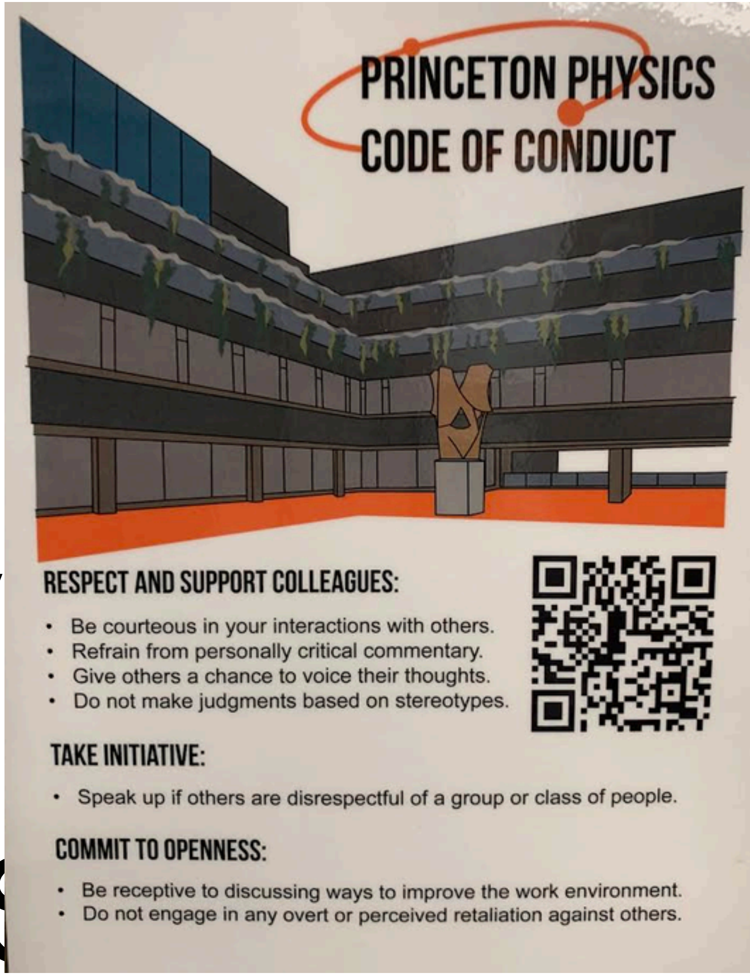
If you see this slide, email the professor with the answer of the square root to 144, plus making a reference to this slide, for bonus points scaled on the order of emails received (earlier email, more points).

- Attendance is required.

  Any absence justifications must be addressed with the office of the appropriate staff .

- Reminder!

We will be courteous, civ
other.
NO SUCH THING AS
OBVIOUS QUESTION
ask ask ask any doubt to clear up.



Integrity – Personal, Academic, Professional.

Presentation and communication.

Determine and describe the reasons the Roman Empire dominated their regions from 27 BC to 476 AC?

The Roman Empire dominated its regions through a combination of overwhelming military superiority, highly organized administrative infrastructure, and strategic cultural assimilation. Professionally trained legions and advanced engineering allowed Rome to conquer and secure vast territories, while an extensive network of roads and centralized governance facilitated efficient trade and communication. Furthermore, by extending citizenship and incorporating local elites into their administrative framework, the Romans successfully fostered regional stability and long-term loyalty across diverse populations.

Presentation and communication.

Determine the values of ‘x’ that satisfy this equation:

$$216x^3 - 8 = 0,$$

$$8\left(27x^3 - 1\right) = 0, \ \ Distributive\ Prop.$$

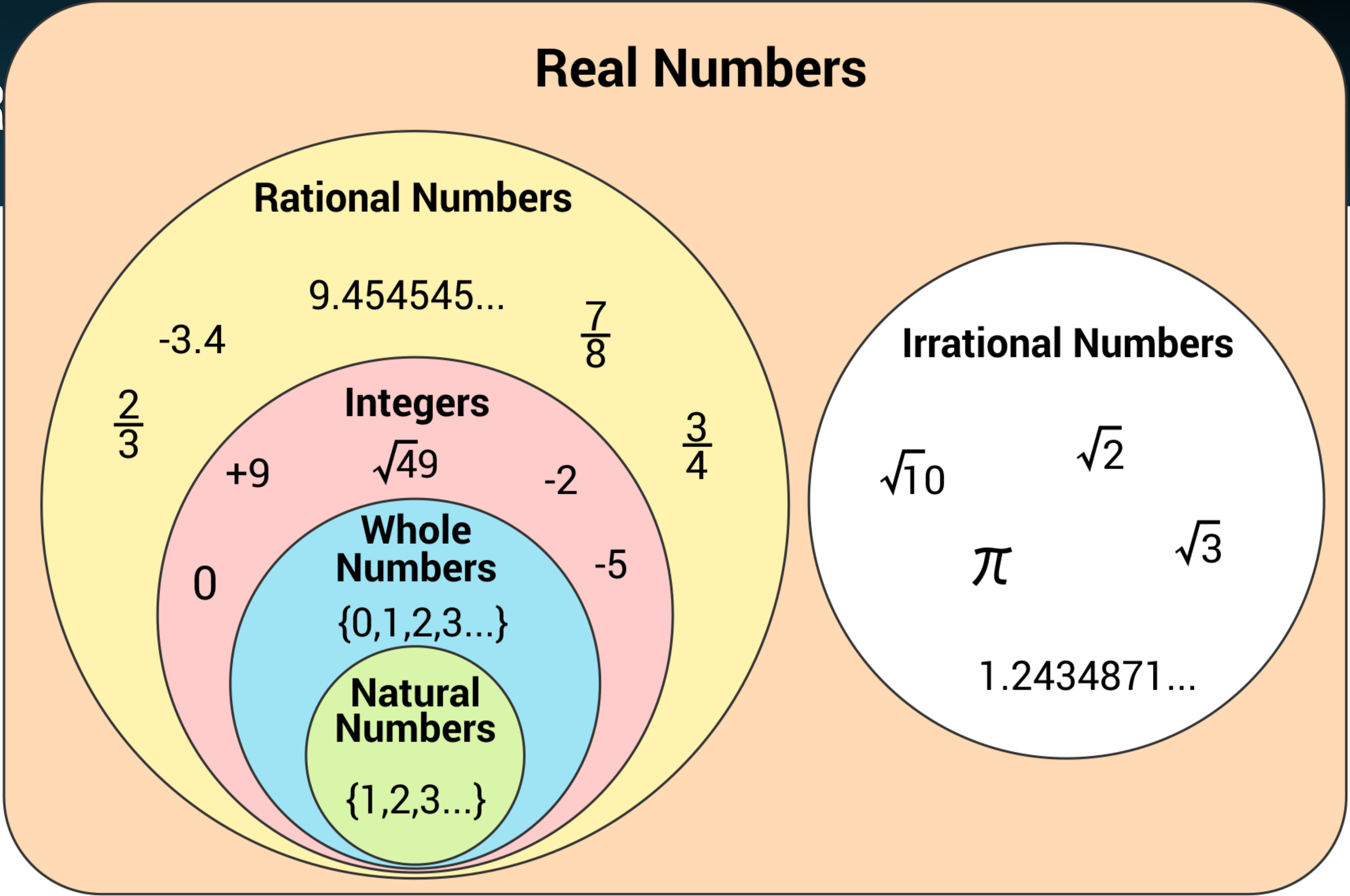
$$\left(27x^3 - 1\right) = 0, \ \ Mult.\ \ eq\ by\ \frac{1}{8}$$

$$(3x-1)\left(3x^2 + 6x + 1\right) = 0, \ \ Polynom.\ \ Factor.$$

$$\implies (3x-1) = 0, \left(3x^2 + 6x + 1\right) = 0, Zero\ Prop.$$

$$\implies x = \{\frac{1}{3}, \frac{\sqrt{6}}{3} - 1, -\frac{\sqrt{6}}{3} - 1\}$$

Review.



Real Numbers
Rational Numbers
9.454545...
-3.4
$\frac{7}{8}$
$\frac{2}{3}$
$\frac{3}{4}$
Integers
+9
$\sqrt{49}$
-2
-5
0
Whole Numbers
{0,1,2,3...}
Natural Numbers
{1,2,3...}
Irrational Numbers
$\sqrt{10}$
$\sqrt{2}$
π
$\sqrt{3}$
1.2434871...

Section 1.1

Properties of the Real Numbers

Name	Addition (A)	Multiplication (M)	Examples
Closure	$a + b$ is a real number	$a \cdot b$ is a real number	A: $1 + \sqrt{2}$ is a real number. M: $2 \cdot \pi$ is a real number.
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$	A: $4 + 7 = 7 + 4$ M: $3 \cdot 8 = 8 \cdot 3$
Associative	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	A: $(2 + 1) + 7 = 2 + (1 + 7)$ M: $(5 \cdot 9) \cdot 13 = 5 \cdot (9 \cdot 13)$
Identity	There is a unique real number 0, called the **additive identity**, such that $a + 0 = a$ and $0 + a = a$.	There is a unique real number 1, called the **multiplicative identity**, such that $a \cdot 1 = a$ and $1 \cdot a = a$.	A: $3 + 0 = 3$ and $0 + 3 = 3$ M: $7 \cdot 1 = 7$ and $1 \cdot 7 = 7$
Inverse	There is a unique real number $-a$, called the **opposite of** $\boldsymbol{a}$, such that $a + (-a) = 0$ and $(-a) + a = 0$.	If $a \neq 0$, there is a unique real number $\frac{1}{a}$, called the **reciprocal of** $\boldsymbol{a}$, such that $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$.	A: $5 + (-5) = 0$ and $(-5) + 5 = 0$ M: $4 \cdot \frac{1}{4} = 1$ and $\frac{1}{4} \cdot 4 = 1$
Distributive	Multiplication distributes over addition.	$a \cdot (b + c) = a \cdot b + a \cdot c$ $(a + b) \cdot c = a \cdot c + b \cdot c$	$3 \cdot (2 + 5) = 3 \cdot 2 + 3 \cdot 5$ $(2 + 5) \cdot 3 = 2 \cdot 3 + 5 \cdot 3$

Section 1.1 Review.

Algebra – fundamental.

Algebra

Properties of Real Numbers

Commutative:	$a + b = b + a$; $ab = ba$
Associative:	$a + (b + c) = (a + b) + c$; $a(bc) = (ab)c$
Additive Identity:	$a + 0 = 0 + a = a$
Additive Inverse:	$-a + a = a + (-a) = 0$
Multiplicative Identity:	$a \cdot 1 = 1 \cdot a = a$
Multiplicative Inverse:	$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1, a \neq 0$
Distributive:	$a(b + c) = ab + ac$

Exponents and Radicals

$a^m \cdot a^n = a^{m+n}$ $\qquad$ $\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$ $\qquad$ $(ab)^m = a^m b^m$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $\qquad$ $a^{-n} = \frac{1}{a^n}$

If n is even, $\sqrt[n]{a^n} = |a|$.

If n is odd, $\sqrt[n]{a^n} = a$.

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \quad a, b \geq 0$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$

Factoring Formulas

$a^2 - b^2 = (a + b)(a - b)$

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Interval Notation

$(a, b) = \{x | a < x < b\}$

$[a, b] = \{x | a \leq x \leq b\}$

$(a, b] = \{x | a < x \leq b\}$

$[a, b) = \{x | a \leq x < b\}$

$(-\infty, a) = \{x | x < a\}$

$(a, \infty) = \{x | x > a\}$

$(-\infty, a] = \{x | x \leq a\}$

$[a, \infty) = \{x | x \geq a\}$

Absolute Value

$|a| \geq 0$

For $a > 0$,

$|X| = a \rightarrow X = -a \quad or \quad X = a,$

$|X| < a \rightarrow -a < X < a,$

$|X| > a \rightarrow X < -a \quad or \quad X > a.$

Section 1.1

SIDE NOTE

You can remember the order of operations by the acronym **PEMDAS** (Please Excuse My Dear Aunt Sally):

Parentheses
Exponents
Multiplication
Division
Addition
Subtraction.

The Order of Operations

Step 1 ▸ Whenever a **fraction bar** is encountered, work separately above and below it.

Step 2 ▸ When **parentheses or other grouping symbols** are present, start inside the innermost pair of grouping symbols and work outward.

Step 3 ▸ Do operations involving **exponents**.

Step 4 ▸ Do **multiplications** and **divisions** in the order in which they occur, **working from left to right**.

Step 5 ▸ Do **additions** and **subtractions** in the order in which they occur, **working from left to right**.

Review.

Worksheet

Mindfulness

Worksheet

Mindfulness

Discussion and
Solutions

Slope–Intercept Form of the Equation of a Line

The **slope–intercept form** of the equation of the line with slope m and y-intercept b is

$$y = mx + b.$$

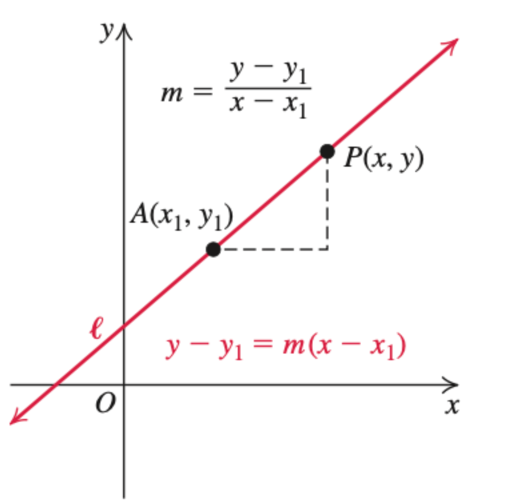


Figure 2.26 ▸ Point–slope form

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The Point–Slope Form of the Equation of a Line

If a line has slope m and passes through the point (x_1, y_1), then the **point–slope form** of an equation of the line is

$$y - y_1 = m(x - x_1).$$

Linear Functions

Let m and b be real numbers. The function $f(x) = mx + b$ is called a **linear function**. If $m = 0$, the function $f(x) = b$ is called a **constant function**. If $m = 1$ and $b = 0$, the resulting function $f(x) = x$ is called the **identity function**. See Figure 2.69.

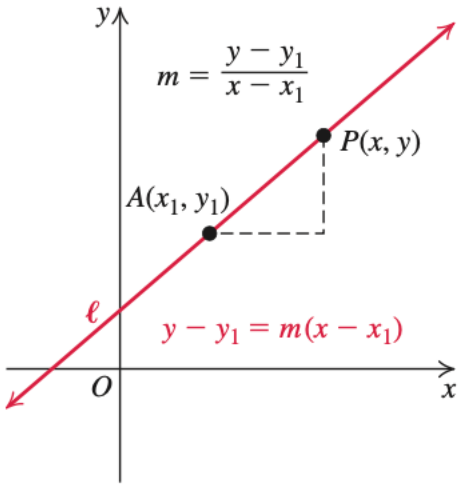

Figure 2.26 ► Point–slope form

Slope–Intercept Form of the Equation of a Line

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The Point–Slope Form of the Equation of a Line

If a line has slope m and passes through the point (x_1, y_1), then the **point–slope form** of an equation of the line is

$$y - y_1 = m(x - x_1).$$

EXAMPLE 2 Finding an Equation of a Line with Given Point and Slope

Find the point–slope form of the equation of the line passing through the point $(1, -2)$ with slope $m = 3$. Then solve for y.

Solution

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point–slope form} \\
y - (-2) &= 3(x - 1) && \text{Substitute } x_1 = 1,\ y_1 = -2, \text{ and } m = 3. \\
y + 2 &= 3x - 3 && \text{Simplify.} \\
y &= 3x - 5 && \text{Solve for } y.
\end{aligned}
$$

Slope

The y-intercept is -5.

P3 sec

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

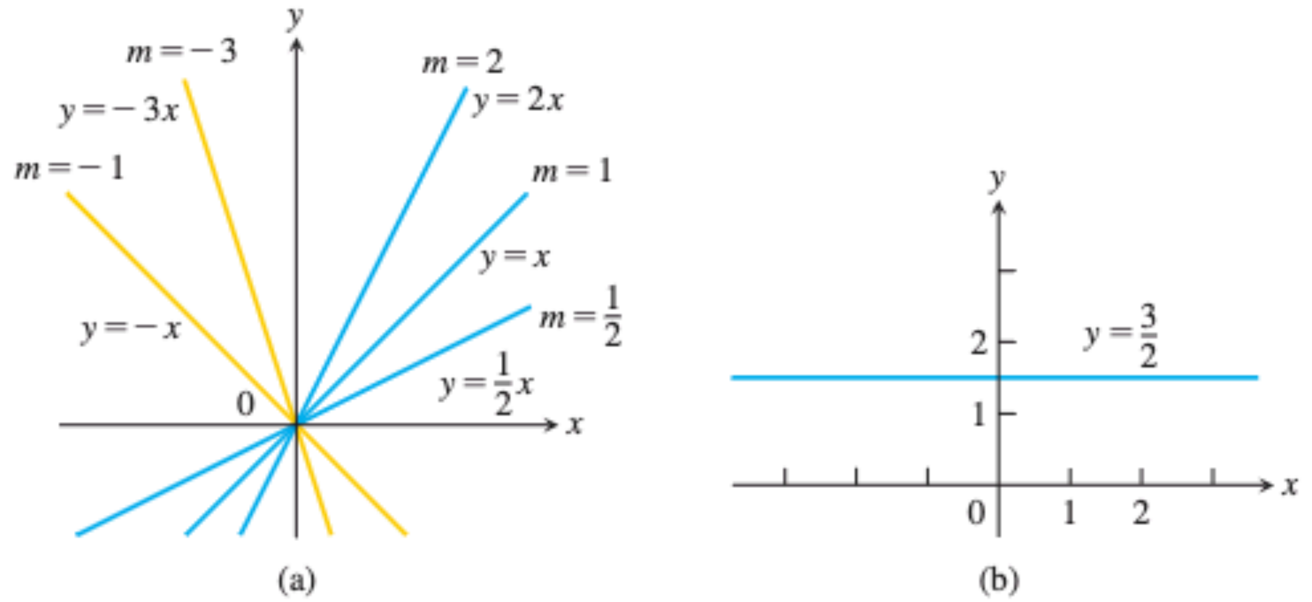

FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k.

P3 sec

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Horizontal and Vertical Lines

An equation of a horizontal line through (h, k) is $\boldsymbol{y = k}$.
An equation of a vertical line through (h, k) is $\boldsymbol{x = h}$.

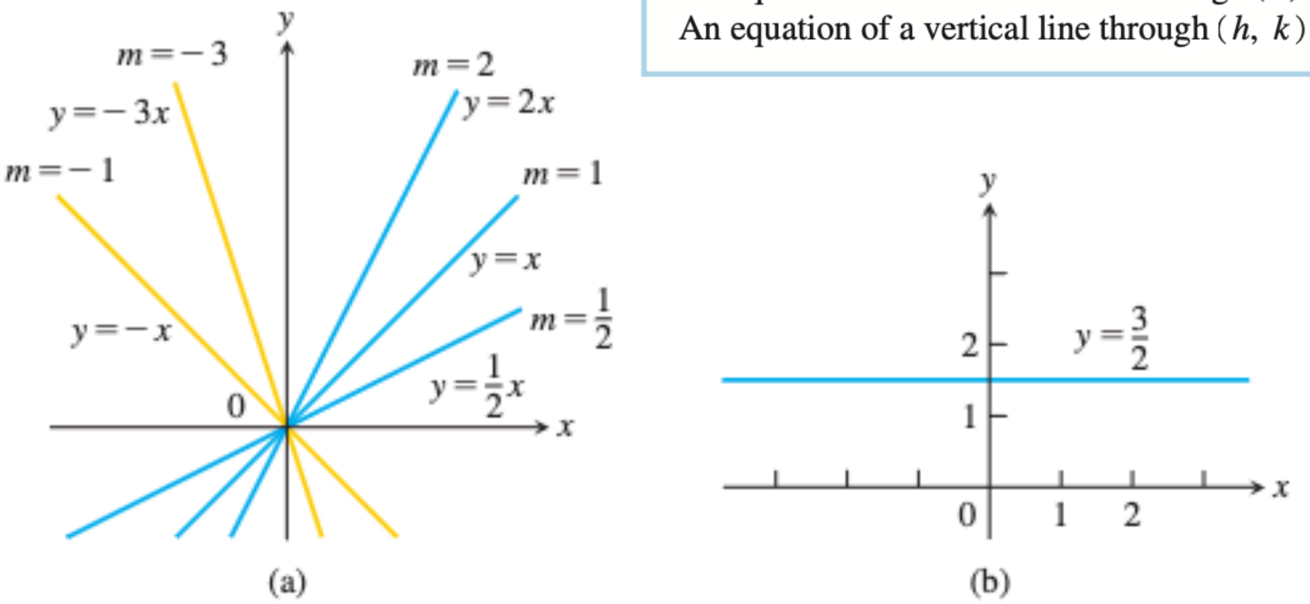


FIGURE 1.14 (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k.

SIDE NOTE

Sometimes it is easier to remember that

(1) given the equation $y = 2$, there is no change in y, because y is constant. The line has to be horizontal $(\Delta y = 0)$.

(2) given the equation $x = 4$, there is no change in x, because x is constant. The line has to be vertical $(\Delta x = 0)$.

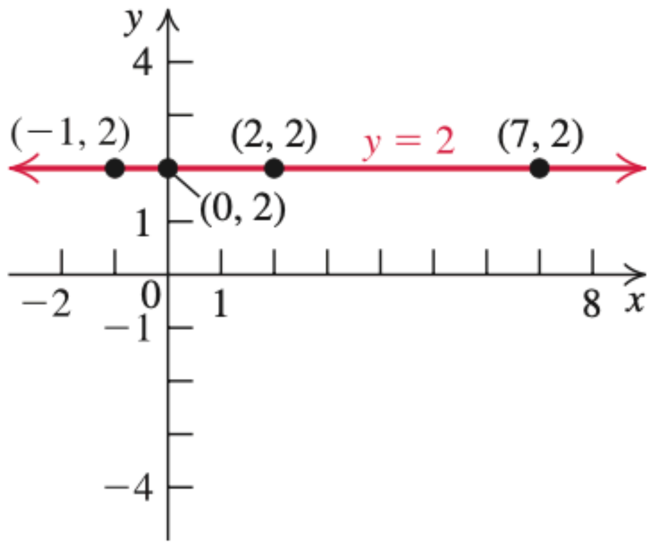


Figure 2.30 ▸ Horizontal line

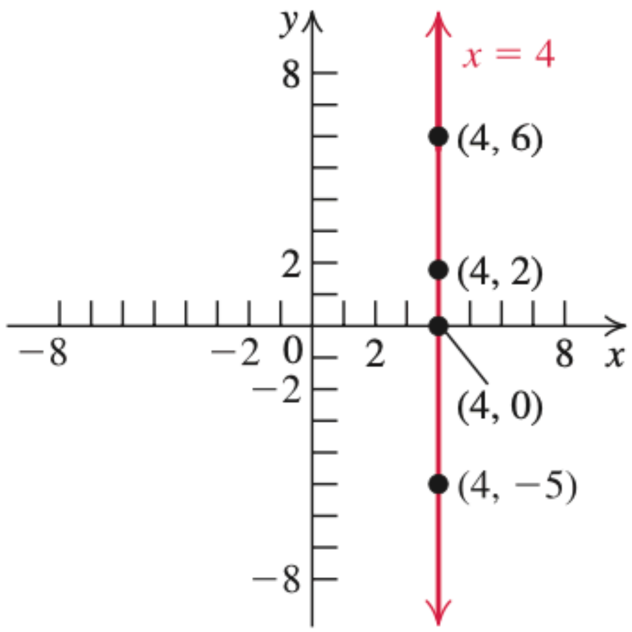


Figure 2.31 ▸ Vertical line

Parallel and Perpendicular Lines

We can use slope to decide whether two lines are parallel, perpendicular, or neither.

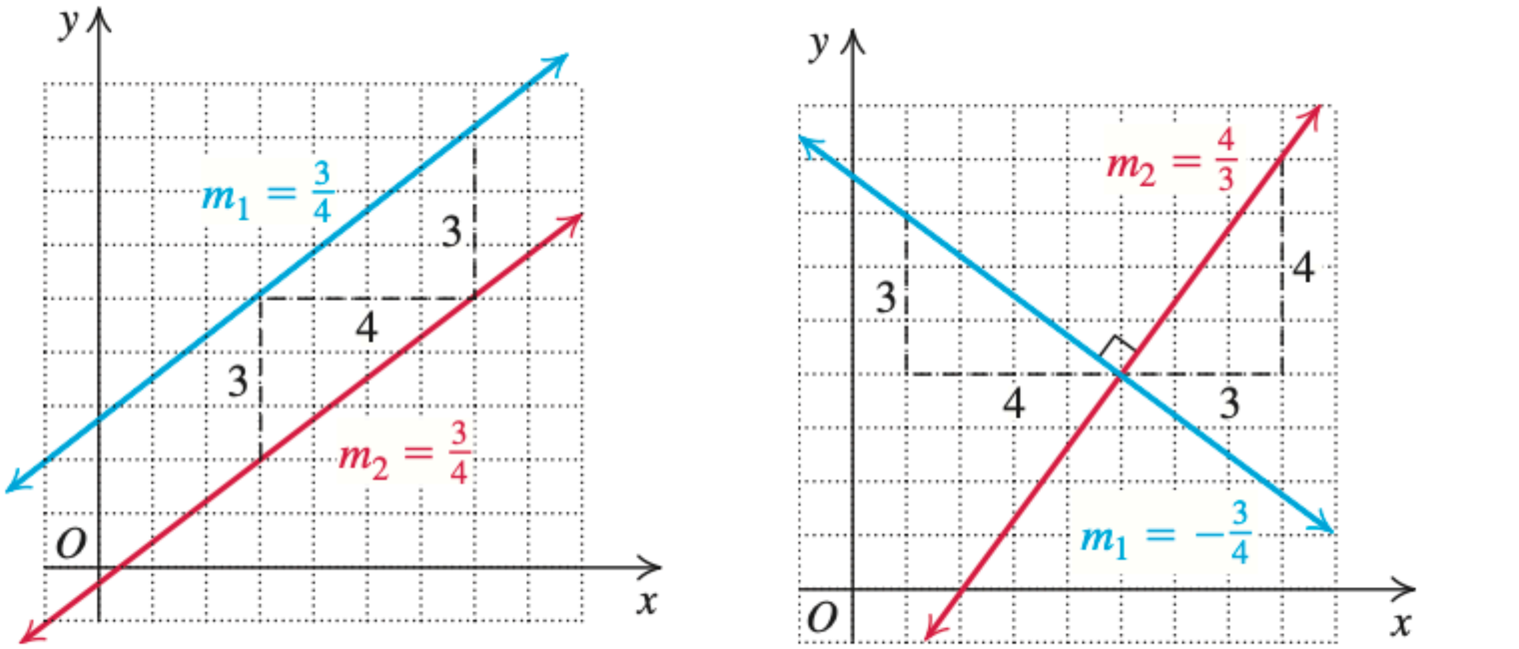
Parallel and Perpendicular Lines

Let ℓ_1 *and* ℓ_2 be two distinct lines with slopes m_1 *and* m_2, respectively. Then

ℓ_1 is parallel to ℓ_2 if and only if $m_1 = m_2$.

ℓ_1 is perpendicular to ℓ_2 if and only if $m_1 \cdot m_2 = -1$.

Any two vertical lines are parallel, as are any two horizontal lines. Any horizontal line is perpendicular to any vertical line.



Parallel lines $m_1 = m_2$

Perpendicular lines $m_1 \cdot m_2 = -1$

Figure 2.33

SUMMARY OF MAIN FACTS

Form	Equation	Slope	Other Information
General	$ax + by + c = 0$ $a \neq 0$, or $b \neq 0$	$-\frac{a}{b}$ if $b \neq 0$ Undefined if $b = 0$	x-intercept is $-\frac{c}{a}$, $a \neq 0$. y-intercept is $-\frac{c}{b}$, $b \neq 0$.
Point–slope	$y - y_1 = m(x - x_1)$	m	Line passes through (x_1, y_1).
Slope–intercept	$y = mx + b$	m	x-intercept is $-\frac{b}{m}$ if $m \neq 0$; y-intercept is b.
Two-intercept	$\frac{x}{a} + \frac{y}{b} = 1$	$-\frac{b}{a}$	x-intercept is a; y-intercept is b.
Vertical line through (h, k)	$x = h$	Undefined	When $h \neq 0$, no y-intercept; x-intercept is h. When $h = 0$, the graph is the y-axis.
Horizontal line through (h, k)	$y = k$	0	When $k \neq 0$, no x-intercept; y-intercept is k. When $k = 0$, the graph is the x-axis.

Parallel and perpendicular lines: Two lines with slopes m_1 and m_2 are (a) parallel if $m_1 = m_2$ and (b) perpendicular if $m_1 \cdot m_2 = -1$.

Solving Linear Equations in One Variable

Linear Equations

A conditional linear equation in one variable, such as x, is an equation that can be written in the **standard form**

$$ax + b = 0,$$

where a and b are real numbers with $a \neq 0$.

SIDE NOTE

When finding the domain of a variable, remember that

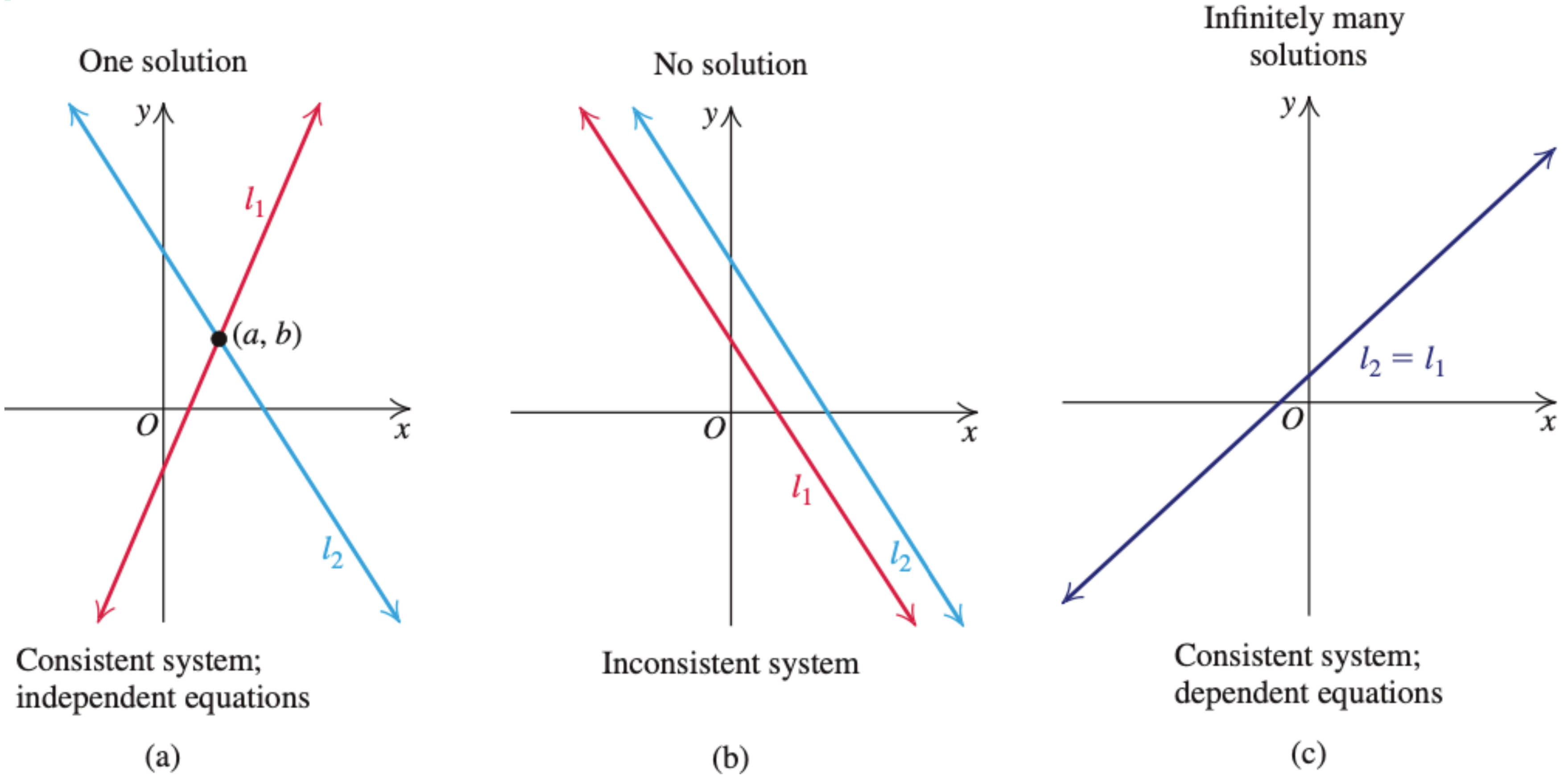
1. Division by 0 is undefined.
2. The square root (or any even root) of a negative number is not a real number.

Identifying Types of Linear Equations

1. An equation that is satisfied by all values in the domain of the variable is an **identity**. For example, $2(x - 1) = 2x - 2$. When you try to solve an identity, you get a true statement that does not depend on the variable, such as $3 = 3$ or $0 = 0$.
2. An equation that is not an identity but is satisfied by at least one number in the domain of the variable is a **conditional** equation. For example, $2x = 6$ is a linear conditional equation; $x = 3$ is a solution, but $x = 0$ is not.
3. An equation that is not satisfied by any value of the variable is an **inconsistent** equation. For example, $x = x + 2$ is an inconsistent equation. Because no number is 2 more than itself, the solution set of the equation $x = x + 2$ is $\varnothing$. When you try to solve an inconsistent equation, you obtain a false statement such as $0 = 2$.

1.1 section

1. One solution, also called **unique solution** (the lines intersect; see Figure 8.2(a)): the system is consistent, and the equations in the system are said to be **independent**.
2. No solution (the lines are parallel; see Figure 8.2(b)): The system is inconsistent.
3. Infinitely many solutions (the lines coincide; see Figure 8.2(c)): The system is consistent, and the equations in the system are said to be **dependent**.



$l_1 \to y = x - 1, l_2 \to y = -x + 2,$ $\quad l_1 \to y = -x, l_2 \to y = -x + 2,$ $\quad l_1 = l_2 := y = x.$

Distance Formula in the Coordinate Plane

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points in the coordinate plane. Then the distance between P and Q, denoted $d(P, Q)$, is given by the **distance formula**

$$\boldsymbol{d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

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The distance between two points in a coordinate plane is the square root of the sum of the square of the difference between their *x*-coordinates and the square of the difference between their *y*-coordinates.

EXAMPLE 3 Finding the Distance Between Two Points

Find the distance between the points $P(-2, 5)$ and $Q(3, -4)$.

Distance Formula in the Coordinate Plane

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EXAMPLE 3 Finding the Distance Between Two Points

Find the distance between the points $P(-2, 5)$ and $Q(3, -4)$.

Solution

Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (3, -4)$. Then

$$x_1 = -2,\ y_1 = 5,\ x_2 = 3, \text{ and } y_2 = -4.$$

$$\begin{aligned} d(P, Q) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\ &= \sqrt{[3 - (-2)]^2 + (-4 - 5)^2} && \text{Substitute the values for } x_1, x_2, y_1, y_2. \\ &= \sqrt{5^2 + (-9)^2} && \text{Simplify.} \\ &= \sqrt{25 + 81} = \sqrt{106} \approx 10.3 && \text{Simplify; use a calculator.} \end{aligned}$$

SIDE NOTE

Remember that in general $\sqrt{a^2 + b^2} \neq a + b$.

Norm (mathematics)

From Simple English Wikipedia, the free encyclopedia

In mathematics, the **norm** of a vector is its length. A **vector** is a mathematical object that has a size, called the *magnitude*, and a direction. For the real numbers, the only norm is the absolute value. For spaces with more dimensions, the norm can be any function p with the following three properties:[1]

1. Scales for real numbers a, that is, $p(ax) = |a|p(x)$.
2. Function of sum is less than sum of functions, that is, $p(x+y) \leq p(x) + p(y)$ (also known as the triangle inequality).
3. $p(x) = 0$ if and only if $x = 0$.

Definition [| change source]

For a vector x, the associated norm is written as $||x||_p$,[2] or Lp where p is some value. The value of the norm of x with some length N is as follows:[3]

$$||x||_p = \sqrt[p]{|x_1|^p + |x_2|^p + \ldots + |x_N|^p}$$

The most common usage of this is the Euclidean norm, also called the standard distance formula.

L^p space

24 languages

In mathematics, the L^p **spaces** are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces. They are sometimes called **Lebesgue spaces**, named after Henri Lebesgue (Dunford & Schwartz 1958, III.3), although according to the Bourbaki group (Bourbaki 1987) they were first introduced by Frigyes Riesz (Riesz 1910).

L^p spaces form an important class of Banach spaces in functional analysis, and of topological vector spaces. Because of their key role in the mathematical analysis of measure and probability spaces, Lebesgue spaces are used also in the theoretical discussion of problems in physics, statistics, economics, finance, engineering, and other disciplines.

Countable Sets

- A set is countable if there is a one-to-one correspondence between the set and $\mathbb{N}$, the natural numbers

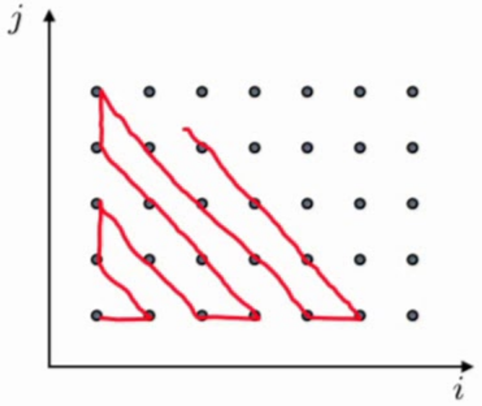
$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & \cdots \\ | & | & | & | & | & \\ 1 & 2 & 3 & 4 & 5 & \cdots \end{array}$$

Countable versus uncountable infinite sets

- Countable: can be put in 1-1 correspondence with positive integers
  - positive integers $1, 2, 3, \ldots$
  - integers $0, 1, -1, 2, -2, 3, -3, \ldots$
  - pairs of positive integers
  - rational numbers q, with $0 < q < 1$

$\frac{1}{2}$.

$\{a_1, a_2, a_3, \ldots\} = \Omega$



Midpoint Formula

The coordinates of the midpoint $M = (x, y)$ on the line segment joining $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are given by

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Functions; Domain and Range

SIDE NOTE

The range of a function from a set X to a set Y does not necessarily have to be the entire set Y.

Function

A **function** from a set X to a set Y is a rule that assigns to each element of X exactly one element of Y. The set X is the **domain** of the function. The set of those elements of Y that correspond (are assigned) to the elements of X is the **range** of the function.

SIDE NOTE

When finding the domain of a variable, remember that

1. Division by 0 is undefined.
2. The square root (or any even root) of a negative number is not a real number.

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f, and y is the **dependent variable** or output value of f at x.

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D.

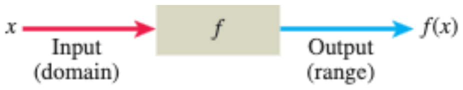


FIGURE 1.1 A diagram showing a function as a kind of machine.

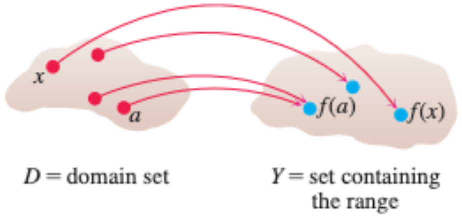


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

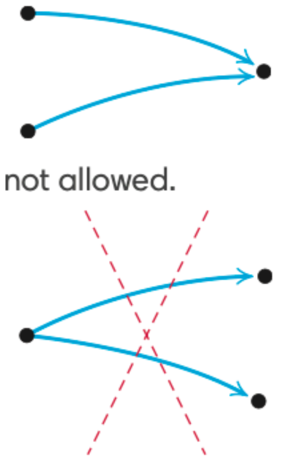
Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

SIDE NOTE

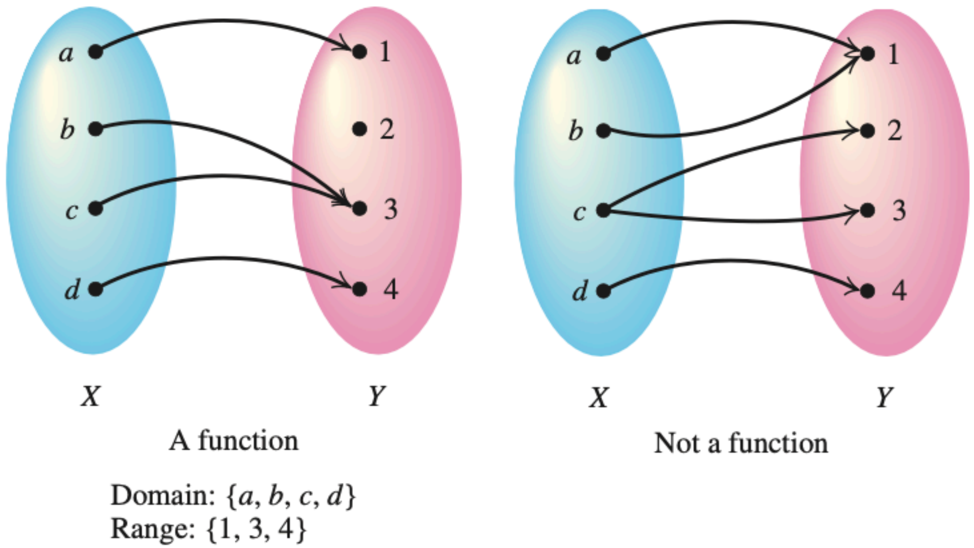
If a correspondence diagram defines a function, then the situation

(a) is allowed.

(b) is not allowed.



To decide whether a correspondence diagram defines a function, we must check whether *any* domain element is paired with more than one range element. If, for some element of X, there are two or more corresponding elements of Y, then the relation is not a function. See the correspondence diagrams in Figure 2.36.



Objective 3 ▸

Graphs of Functions

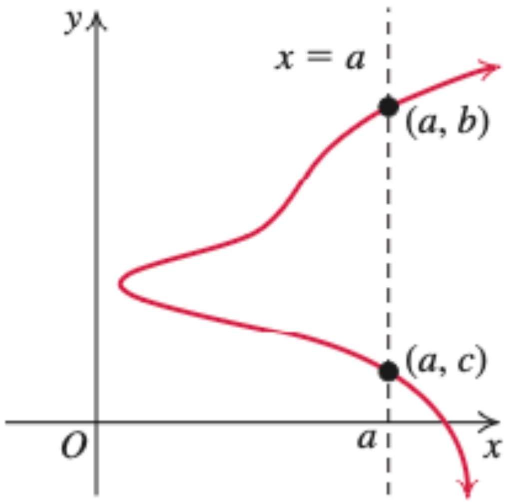


Figure 2.44 ▸ Graph does not represent a function.

The *graph of a function f* is the set of ordered pairs $(x, f(x))$ such that x is in the domain of f. That is, the graph of f is the graph of the equation $y = f(x)$. We sketch the graph of $y = f(x)$ by plotting points and joining them with a smooth curve. The graph of a function provides valuable visual information about the function.

Figure 2.44 shows that not every curve in the plane is the graph of a function. In this figure, a vertical line $x = a$ intersects the curve at two distinct points (a, b) and (a, c). This curve cannot be the graph of $y = f(x)$ for any function f because having $f(a) = b$ and $f(a) = c$ means that f assigns two different range values to the same domain element a. We express this statement, called the vertical-line test, as follows:

Vertical-Line Test

If no vertical line intersects the graph of a relation at more than one point, then the graph of the relation is the graph of a function.

EXAMPLE 2 Determining Whether an Equation Defines a Function

In each equation, determine whether y is a function of x.

a. $6x^2 - 3y = 12$ **b.** $y^2 - x^2 = 4$

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In each equation, determine whether y is a function of x.

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Solution

Solve each equation for y in terms of x. If more than one value of y corresponds to the same value of x, then y is not a function of x.

a.

$$\begin{aligned} 6x^2 - 3y &= 12 && \text{Original equation} \\ 6x^2 - 3y + 3y - 12 &= 12 + 3y - 12 && \text{Add } 3y - 12 \text{ to both sides.} \\ 6x^2 - 12 &= 3y && \text{Simplify.} \\ 2x^2 - 4 &= y && \text{Divide by 3.} \\ y &= 2x^2 - 4 && \text{Interchange sides.} \end{aligned}$$

The last equation shows that only one value of y corresponds to each value of x. For example, if $x = 0$, then $y = 0 - 4 = -4$. So y is a function of x.

EXAMPLE 2 Determining Whether an Equation Defines a Function

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Solve each equation for y in terms of x. If more than one value of y corresponds to the same value of x, then y is not a function of x.

a.

$6x^2 - 3y = 12$	Original equation
$6x^2 - 3y + 3y - 12 = 12 + 3y - 12$	Add $3y - 12$ to both sides.
$6x^2 - 12 = 3y$	Simplify.
$2x^2 - 4 = y$	Divide by 3.
$y = 2x^2 - 4$	Interchange sides.

The last equation shows that only one value of y corresponds to each value of x. For example, if $x = 0$, then $y = 0 - 4 = -4$. So y is a function of x.

b.

$y^2 - x^2 = 4$	Original equation
$y^2 - x^2 + x^2 = 4 + x^2$	Add x^2 to both sides.
$y^2 = x^2 + 4$	Simplify.
$y = \pm\sqrt{x^2 + 4}$	Square root property

The last equation shows that two values of y correspond to each value of x. For example, if $x = 0$, then $y = \pm\sqrt{0^2 + 4} = \pm\sqrt{4} = \pm 2$. Both $y = 2$ and $y = -2$ correspond to $x = 0$. Therefore, y is not a function of x. However, the equation $y^2 - x^2 = 4$ defines two functions:

$f_1(x) = \sqrt{x^2 + 4}$; domain: $(-\infty, \infty)$; range: $[2, \infty)$, and
$f_2(x) = -\sqrt{x^2 + 4}$; domain: $(-\infty, \infty)$; range $(-\infty, -2]$. See Figure 2.42.

Graph of an Equation

The **graph of an equation** in two variables, such as $\boldsymbol{x}$ and $\boldsymbol{y}$, is the set of all ordered pairs $(a,\ b)$ in the coordinate plane that satisfy the equation.

Graph of an Equation

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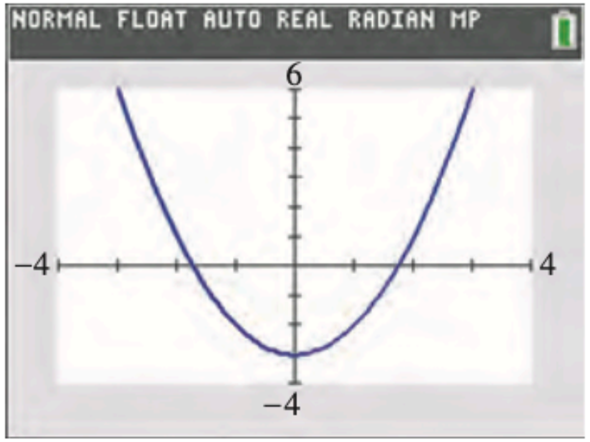
TECHNOLOGY CONNECTION

Calculator graph of $y = x^2 - 3$



EXAMPLE 1 Sketching a Graph by Plotting Points

Sketch the graph of $y = x^2 - 3$.

Solution

There are infinitely many solutions of the equation $y = x^2 - 3$. To find a few, we choose integer values of x between -3 and 3. Then we find the corresponding values of y as shown in Table 2.3.

TABLE 2.3

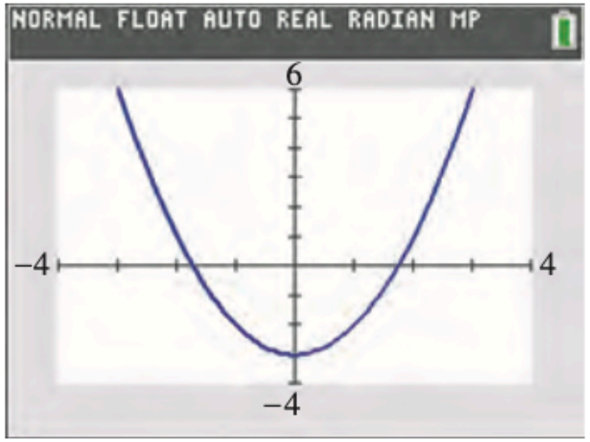
x	$y = x^2 - 3$	(x, y)
-3	$y = (-3)^2 - 3 = 9 - 3 = 6$	$(-3, 6)$
-2	$y = (-2)^2 - 3 = 4 - 3 = 1$	$(-2, 1)$
-1	$y = (-1)^2 - 3 = 1 - 3 = -2$	$(-1, -2)$
0	$y = 0^2 - 3 = 0 - 3 = -3$	$(0, -3)$
1	$y = 1^2 - 3 = 1 - 3 = -2$	$(1, -2)$
2	$y = 2^2 - 3 = 4 - 3 = 1$	$(2, 1)$
3	$y = 3^2 - 3 = 9 - 3 = 6$	$(3, 6)$

Graph of an Equation

The **graph of an equation** in two variables, such as x and y, is the set of all ordered pairs (a, b) in the coordinate plane that satisfy the equation.

TECHNOLOGY CONNECTION

Calculator graph of $y = x^2 - 3$



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[illegible]	[illegible] $0^2 - 3 = 0 - 3 = -3$	$(0, -3)$
[illegible]	[illegible] $1^2 - 3 = 1 - 3 = -2$	$(1, -2)$
[illegible]	[illegible] $2^2 - 3 = 4 - 3 = 1$	$(2, 1)$
[illegible]	[illegible] $3^2 - 3 = 9 - 3 = 6$	$(3, 6)$

Sketching a Graph by Plotting Points

Step 1 ▸ Make a representative table of solutions of the equation.

Step 2 ▸ Plot the solutions as ordered pairs in the coordinate plane.

Step 3 ▸ Connect the solutions in Step 2 with a smooth curve.

Rules for Solving Absolute Value Inequalities

If $a > 0$ and u is an algebraic expression, then

1. $|u| < a$ is equivalent to $-a < u < a$, or u in $(-a, a)$.
2. $|u| \leq a$ is equivalent to $-a \leq u \leq a$, or u in $[-a, a]$.
3. $|u| > a$ is equivalent to $u < -a$ or $u > a$, or u in $(-\infty, -a) \cup (a, \infty)$.
4. $|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$, or u in $(-\infty, -a] \cup [a, \infty)$.

Rules for Solving Absolute Value Inequalities

If $a > 0$ and u is an algebraic expression, then

1. $|u| < a$ is equivalent to $-a < u < a$, or u in $(-a, a)$.
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3. $|u| > a$ is equivalent to $u < -a$ or $u > a$, or u in $(-\infty, -a) \cup (a, \infty)$.
4. $|u| \ge a$ is equivalent to $u \le -a$ or $u \ge a$, or u in $(-\infty, -a] \cup [a, \infty)$.

Graphing Piecewise Functions

Let's consider how to graph a piecewise function. The **absolute value function**, $f(x) = |x|$, can be expressed as a piecewise function by using the definition of absolute value:

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}.$$

The first line in the function means that if $x < 0$, we use the equation $y = f(x) = -x$. So, if $x = -3$, then

$$\begin{aligned} y &= f(-3) = -(-3) && \text{Substitute } -3 \text{ for } x \text{ in } f(x) = -x. \\ &= 3. && \text{Simplify.} \end{aligned}$$

Thus, $(-3, 3)$ is a point on the graph of $y = |x|$. However, if $x \ge 0$, we use the second line in the function, which is $y = f(x) = x$. So, if $x = 2$, then

$$y = f(2) = 2. \quad \text{Substitute 2 for } x \text{ in } f(x) = x.$$

So $(2, 2)$ is a point on the graph of $y = |x|$. The two pieces $y = -x$ and $y = x$ are linear functions. We graph the appropriate parts of these lines ($y = -x$ for $x < 0$ and $y = x$ for $x \ge 0$) to form the graph of $y = |x|$. See Figure 2.75.

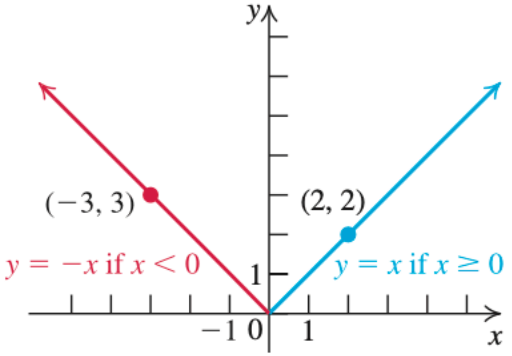


Figure 2.75 ▸ Graph of $y = |x|$

Objective 2 ▸

Intercepts

We examine the points where a graph intersects (crosses or touches) the coordinate axes. Because all points on the x-axis have a y-coordinate of 0, any point where a graph intersects the x-axis has the form $(a, 0)$. See Figure 2.13. The number a is called an ***x*-intercept** of the graph. Similarly, any point where a graph intersects the y-axis has the form $(0, b)$, and the number b is called a ***y*-intercept** of the graph.

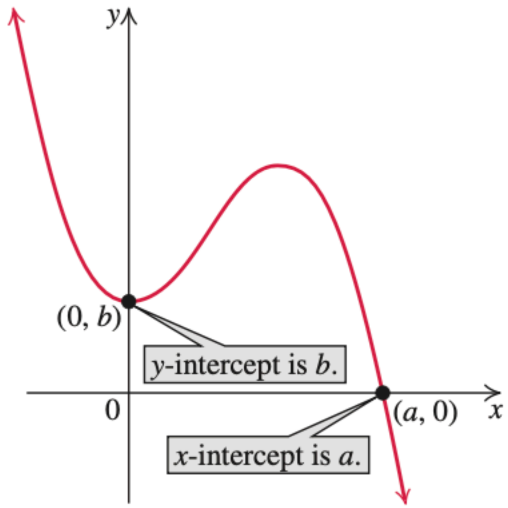


Figure 2.13 ▸ Intercepts of a graph

Finding the Intercepts of the Graph of an Equation

Step 1 ▸ To find the x-intercepts of the graph of an equation, set $y = 0$ in the equation and solve for x.

Step 2 ▸ To find the y-intercepts of the graph of an equation, set $x = 0$ in the equation and solve for y.

Note that only real number solutions in Steps 1 and 2 correspond to the intercepts.

SUMMARY OF MAIN FACTS

A function is usually described in one or more of the following six ways:

- A correspondence diagram
- A set of ordered pairs
- A table of values
- An equation or a formula
- A scatterplot or a graph
- In the words *y is a function of x*

Input	Output
x	y or $f(x)$
First coordinate	Second coordinate
Independent variable	Dependent variable
Domain is the set of all inputs.	Range is the set of all outputs.

- Find domain, range, x intercept, y intercept, and graph the following:

23. $y = |x| + 1$

24. $y = -|x| + 1$

25. $y = 4 - x^2$

26. $y = x^2 - 4$

27. $y = \sqrt{9 - x^2}$

28. $y = -\sqrt{9 - x^2}$

29. $y = x^3$

30. $y = -x^3$

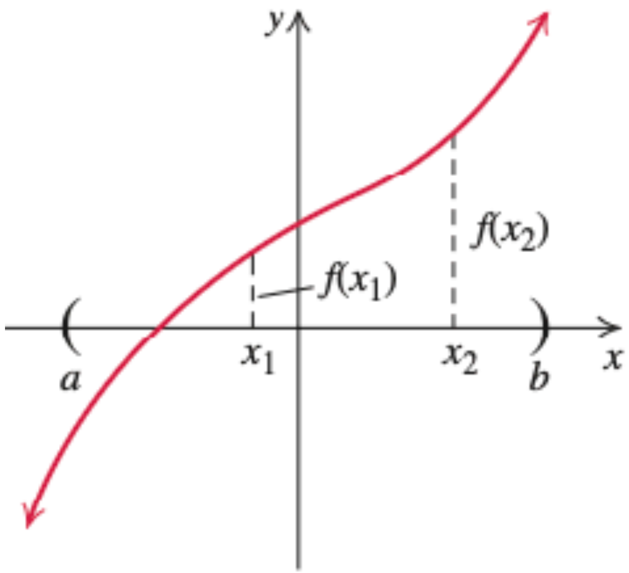
Increasing, Decreasing, and Constant Functions

Let f be a function and let x_1 and x_2 be any two numbers in an open interval (a, b) contained in the domain of f. See Figure 2.58. The symbols a and b may represent real numbers, $-\infty$, or ∞. Then

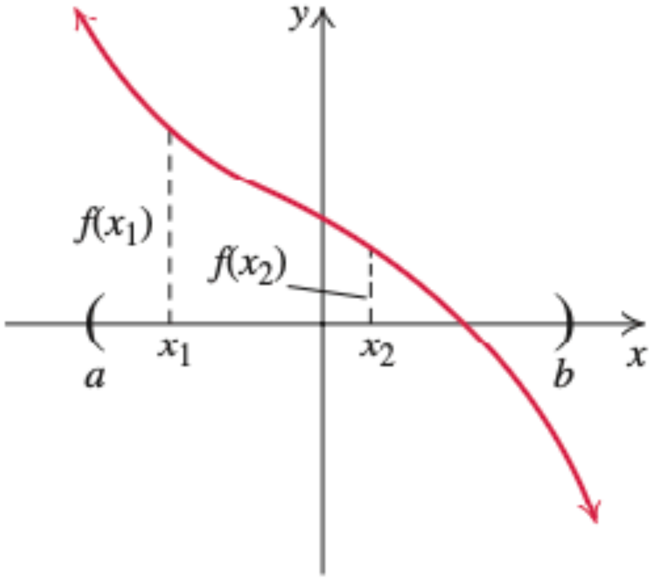
(i) f is called an **increasing function** on (a, b) if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

(ii) f is called a **decreasing function** on (a, b) if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

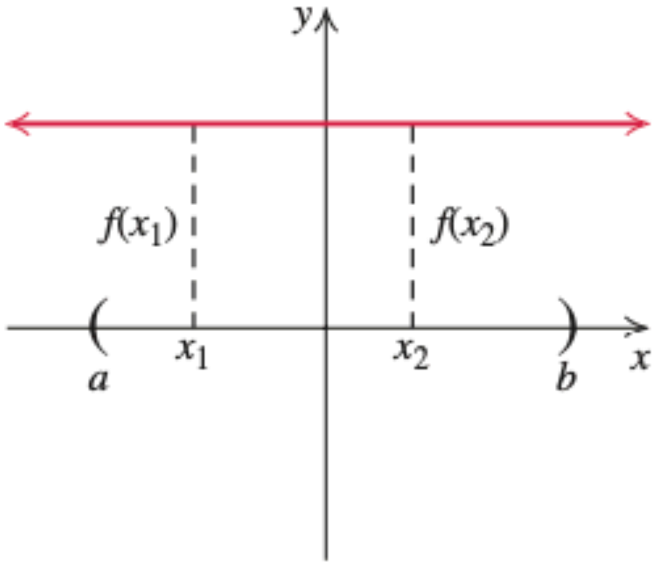
(iii) f is **constant** on (a, b) if $x_1 < x_2$ implies $f(x_1) = f(x_2)$.



Increasing function on (a, b)
(a)



Decreasing function on (a, b)
(b)



Constant function on (a, b)
(c)

Relative Maximum and Relative Minimum

If a is in the domain of a function f, we say that the value $f(a)$ is a **relative maximum of f** if there is an interval (x_1, x_2) containing a such that

$$f(a) \geq f(x) \text{ for every } x \text{ in the interval } (x_1, x_2).$$

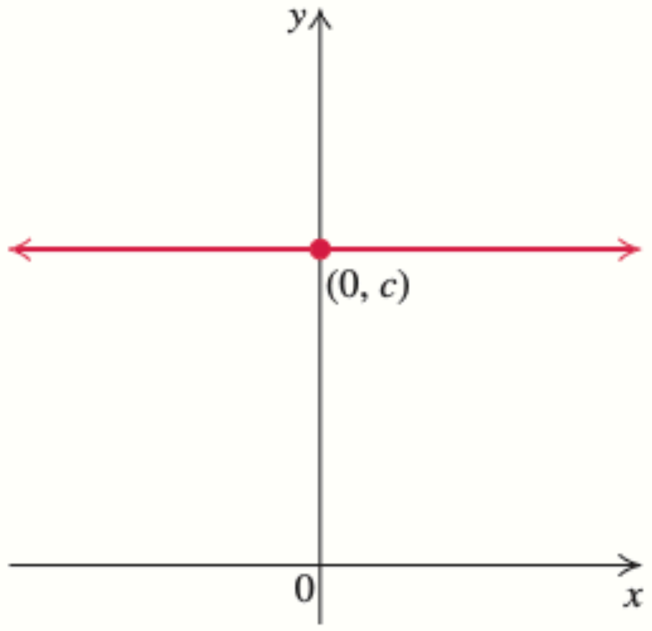
We say that the value $f(a)$ is a **relative minimum of f** if there is an interval (x_1, x_2) containing a such that

$$f(a) \leq f(x) \text{ for every } x \text{ in the interval } (x_1, x_2).$$

A Library of Basic Functions

Constant Function

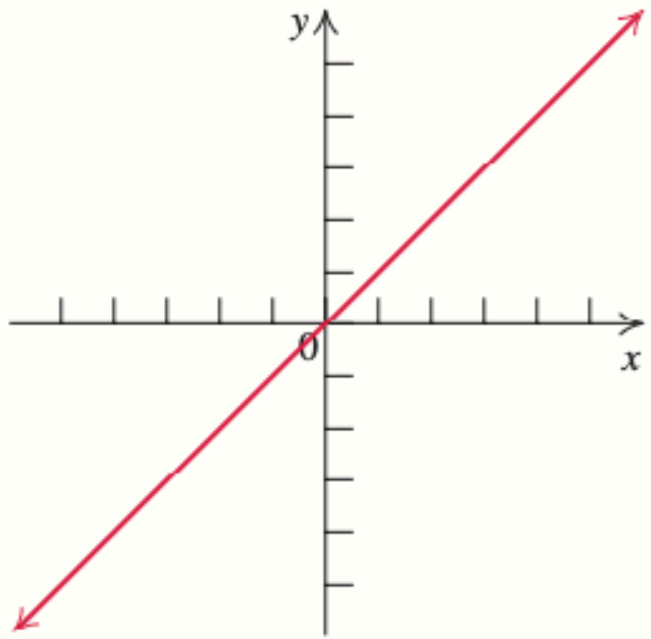
$f(x) = c$



Domain: $(-\infty, \infty)$
Range: $\{c\}$
Constant on $(-\infty, \infty)$
Even function (y-axis symmetry)

Identity Function

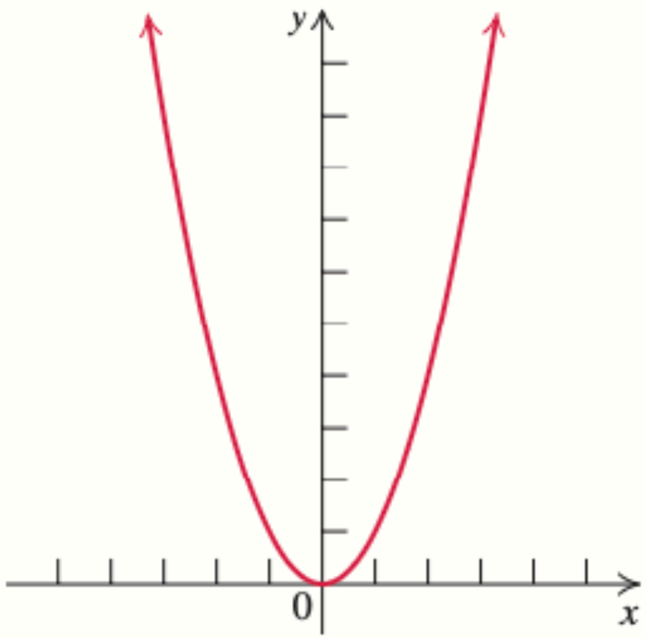
$f(x) = x$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Increasing on $(-\infty, \infty)$
Odd function (origin symmetry)

Squaring Function

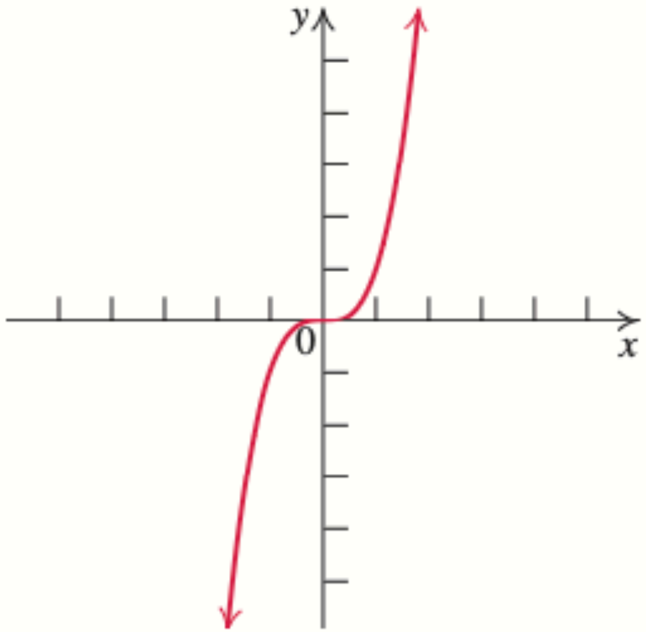
$f(x) = x^2$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function (y-axis symmetry)

Cubing Function

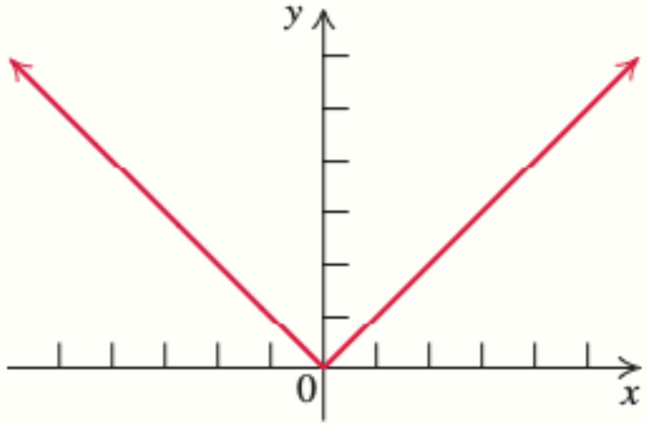
$f(x) = x^3$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Increasing on $(-\infty, \infty)$
Odd function (origin symmetry)

Absolute Value Function

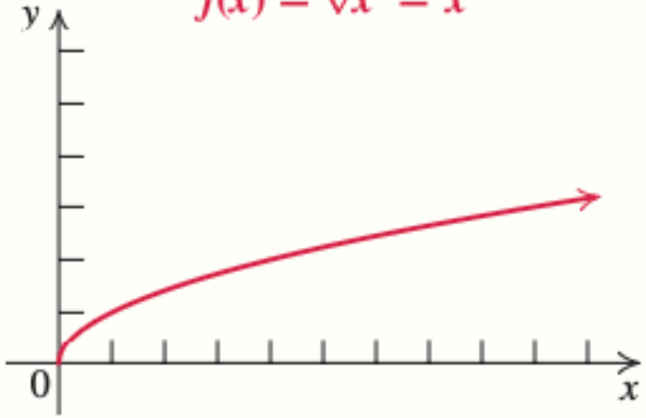
$f(x) = |x|$



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function (y-axis symmetry)

Square Root Function

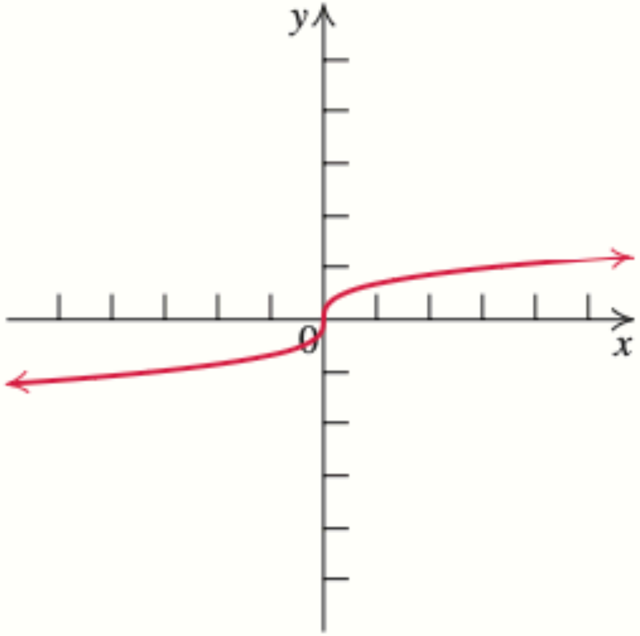
$f(x) = \sqrt{x} = x^{1/2}$



Domain: $[0, \infty)$
Range: $[0, \infty)$
Increasing on $(0, \infty)$
Neither even nor odd (no symmetry)

Cube Root Function

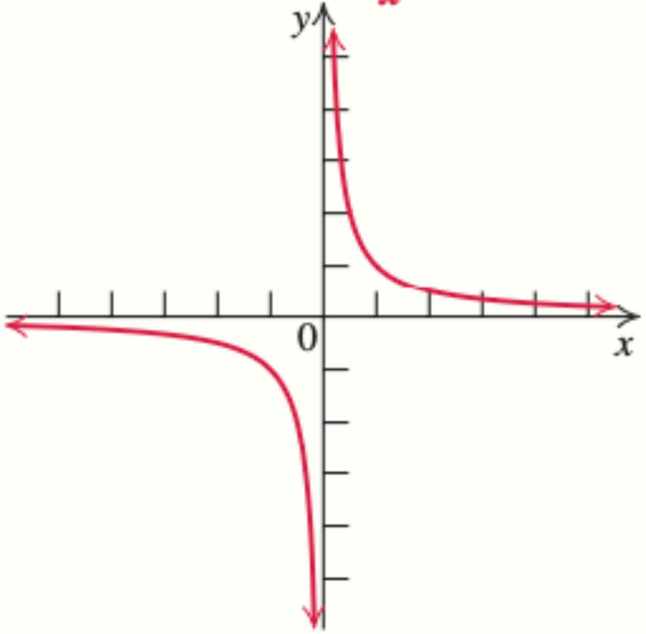
$f(x) = \sqrt[3]{x} = x^{1/3}$



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Increasing on $(-\infty, \infty)$
Odd function (origin symmetry)

Reciprocal Function

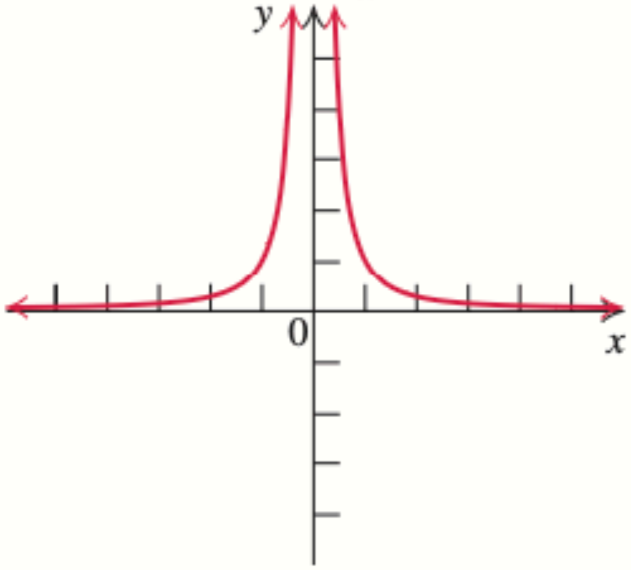
$f(x) = \dfrac{1}{x}$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Decreasing on $(-\infty, 0) \cup (0, \infty)$
Odd function (origin symmetry)

Reciprocal Square Function

$f(x) = \dfrac{1}{x^2}$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$
Even function (y-axis symmetry)

Composition of Functions

Composition of Functions

Algebraic operations

Composition of Functions

Algebraic operations

For two functions $f\left(x\right)$ and $g\left(x\right)$ with real number outputs, we define new functions $f+g,\ \ f-g,\ \ fg$, and $\frac{f}{g}$ by the relations

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(fg)(x) &= f(x)g(x)\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}
\end{aligned}
$$

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions. The **sum** $f + g$, the **difference** $f - g$, the **product** fg, and the **quotient** $\frac{f}{g}$ are functions defined as follows:

- Sum $\quad (f + g)(x) = f(x) + g(x)$
- Difference $\quad (f - g)(x) = f(x) - g(x)$
- Product $\quad fg(x) = f(x) \cdot g(x)$; also written as $(fg)(x)$
- Quotient $\quad \frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided that $g(x) \neq 0$; also written as $\left(\frac{f}{g}\right)(x)$

The **domain** of each of the new functions consists of those values of x that are common to the domains of f and g, except that for $\frac{f}{g}$ all x for which $g(x) = 0$ must be excluded.

WARNING

When rewriting a function such as

$$\left(\frac{f}{g}\right)(x) = \frac{(x-2)(x-4)}{x-2}, \quad x \neq 2,$$

by dividing out the common factor, remember to specify "$x \neq 2$."

We identify the domain before we simplify and write

$$\left(\frac{f}{g}\right)(x) = x - 4, \quad x \neq 2.$$

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$

b. $\left(\frac{f}{g}\right)(3)$

c. $(f + g)(x)$

d. $(f - g)(x)$

e. $(fg)(x)$

f. $\left(\frac{f}{g}\right)(x)$

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\dfrac{f}{g}\right)(3)$

c. $(f + g)(x)$ **d.** $(f - g)(x)$

e. $(fg)(x)$ **f.** $\left(\dfrac{f}{g}\right)(x)$

Solution

a.

$$f(4) = 4^2 - 6(4) + 8 = 0 \qquad \text{Replace } x \text{ with 4 in } f(x).$$

$$g(4) = 4 - 2 = 2 \qquad \text{Replace } x \text{ with 4 in } g(x).$$

$$(f - g)(4) = f(4) - g(4) \qquad \text{Definition of difference}$$

$$= 0 - 2 = -2 \qquad \text{Replace } f(4) \text{ with 0, } g(4) \text{ with 2.}$$

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\frac{f}{g}\right)(3)$

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Solution

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$$f(4) = 4^2 - 6(4) + 8 = 0 \qquad \text{Replace } x \text{ with 4 in } f(x).$$

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$$(f - g)(4) = f(4) - g(4) \qquad \text{Definition of difference}$$

$$= 0 - 2 = -2 \qquad \text{Replace } f(4) \text{ with 0, } g(4) \text{ with 2.}$$

b.

$$f(3) = 3^2 - 6(3) + 8 = -1 \qquad \text{Replace } x \text{ with 3 in } f(x).$$

$$g(3) = 3 - 2 = 1 \qquad \text{Replace } x \text{ with 3 in } g(x).$$

$$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{-1}{1} = -1 \qquad \text{Replace } f(3) \text{ with } -1, g(3) \text{ with 1.}$$

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\dfrac{f}{g}\right)(3)$

c. $(f + g)(x)$ **d.** $(f - g)(x)$

e. $(fg)(x)$ **f.** $\left(\dfrac{f}{g}\right)(x)$

Solution

a.

$$f(4) = 4^2 - 6(4) + 8 = 0 \qquad \text{Replace } x \text{ with 4 in } f(x).$$

$$g(4) = 4 - 2 = 2 \qquad \text{Replace } x \text{ with 4 in } g(x).$$

$$(f - g)(4) = f(4) - g(4) \qquad \text{Definition of difference}$$

$$= 0 - 2 = -2 \qquad \text{Replace } f(4) \text{ with 0, } g(4) \text{ with 2.}$$

b.

$$f(3) = 3^2 - 6(3) + 8 = -1 \qquad \text{Replace } x \text{ with 3 in } f(x).$$

$$g(3) = 3 - 2 = 1 \qquad \text{Replace } x \text{ with 3 in } g(x).$$

$$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{-1}{1} = -1 \qquad \text{Replace } f(3) \text{ with } -1,\ g(3) \text{ with 1.}$$

c.

$$(f + g)(x) = f(x) + g(x) \qquad \text{Definition of sum}$$

$$= (x^2 - 6x + 8) + (x - 2) \qquad \text{Add } f(x) \text{ and } g(x).$$

$$= x^2 - 5x + 6 \qquad \text{Combine like terms.}$$

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\dfrac{f}{g}\right)(3)$

c. $(f + g)(x)$ **d.** $(f - g)(x)$

e. $(fg)(x)$ **f.** $\left(\dfrac{f}{g}\right)(x)$

Solution

a.

$f(4) = 4^2 - 6(4) + 8 = 0$	Replace x with 4 in $f(x)$.
$g(4) = 4 - 2 = 2$	Replace x with 4 in $g(x)$.
$(f - g)(4) = f(4) - g(4)$	Definition of difference
$= 0 - 2 = -2$	Replace $f(4)$ with 0, $g(4)$ with 2.

b.

$f(3) = 3^2 - 6(3) + 8 = -1$	Replace x with 3 in $f(x)$.
$g(3) = 3 - 2 = 1$	Replace x with 3 in $g(x)$.
$\left(\dfrac{f}{g}\right)(3) = \dfrac{f(3)}{g(3)} = \dfrac{-1}{1} = -1$	Replace $f(3)$ with -1, $g(3)$ with 1.

c.

$(f + g)(x) = f(x) + g(x)$	Definition of sum
$= (x^2 - 6x + 8) + (x - 2)$	Add $f(x)$ and $g(x)$.
$= x^2 - 5x + 6$	Combine like terms.

d.

$(f - g)(x) = f(x) - g(x)$	Definition of difference
$= (x^2 - 6x + 8) + (x - 2)$	Subtract $g(x)$ from $f(x)$.
$= x^2 - 7x + 10$	Combine like terms.

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\frac{f}{g}\right)(3)$

c. $(f + g)(x)$ **d.** $(f - g)(x)$

e. $(fg)(x)$ **f.** $\left(\frac{f}{g}\right)(x)$

e.

$(fg)(x) = f(x) \cdot g(x)$	Definition of product
$= (x^2 - 6x + 8)(x - 2)$	Multiply $f(x)$ and $g(x)$.
$= x^2(x - 2) - 6x(x - 2) + 8(x - 2)$	Distributive property
$= x^3 - 2x^2 - 6x^2 + 12x + 8x - 16$	Distributive property
$= x^3 - 8x^2 + 20x - 16$	Combine like terms.

EXAMPLE 1 Combining Functions

Let $f(x) = x^2 - 6x + 8$ and $g(x) = x - 2$. Find each of the following functions or function values.

a. $(f - g)(4)$ **b.** $\left(\frac{f}{g}\right)(3)$

c. $(f + g)(x)$ **d.** $(f - g)(x)$

e. $(fg)(x)$ **f.** $\left(\frac{f}{g}\right)(x)$

f. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$ Definition of quotient

$= \frac{x^2 - 6x + 8}{x - 2}, \quad x - 2 \neq 0$ Replace $f(x)$ with $x^2 - 6x + 8$, $g(x)$ with $x - 2$.

$= \frac{(x - 2)(x - 4)}{x - 2}, \quad x \neq 2$ Factor the numerator.

$= x - 4, \quad x \neq 2$

Because f and g are polynomials, the domain of both f and g is the set of all real numbers, or in interval notation, $(-\infty, \infty)$. So the domain for $f + g$, $f - g$, and fg is $(-\infty, \infty)$. However, for $\frac{f}{g}$, we must exclude 2 because $g(2) = 0$. The domain for $\frac{f}{g}$ is the set of all real numbers x, $x \neq 2$, or in interval notation, $(-\infty, 2) \cup (2, \infty)$.

Composition of Functions

Algebraic operations

EXAMPLE 1

Performing Algebraic Operations on Functions

Find and simplify the functions $(g - f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x - 1$ and $g(x) = x^2 - 1$. Are they the same function?

Composition of Functions

Algebraic

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ (g-f)(x) &= x^2-x\\ (g-f)(x) &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ \left(\frac{g}{f}\right)(x) &= \frac{(x+1)(x-1)}{x-1} && \text{where} x \neq 1\\ \left(\frac{g}{f}\right)(x) &= x+1\end{aligned}$$

Composition of Functions

Algebraic

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ (g-f)(x) &= x^2-x\\ (g-f)(x) &= x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ \left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ \left(\frac{g}{f}\right)(x) &= \frac{(x+1)(x-1)}{x-1} && \text{where} x\neq 1\\ \left(\frac{g}{f}\right)(x) &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x\neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Suggested practice

Practice Problem 1. Let $f(x) = 3x - 1$ and $g(x) = x^2 + 2$. Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$. ■

Compositio

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions**.** The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x ," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

g(*x*), the output of *g*
is the input of *f*

$$(f \circ g)(x) = f(\overline{g(\underline{x})})$$

x is the input of *g*

Composition of Functions

Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

Composition of Functions

Determining whether Composition of Functions is Commutative

Using the functions provided, find $f\left(g\left(x\right)\right)$ and $g\left(f\left(x\right)\right)$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

Solution

Let's begin by substituting $g\left(x\right)$ into $f\left(x\right)$.

$$\begin{aligned} f(g(x)) &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute $f\left(x\right)$ into $g\left(x\right)$.

$$\begin{aligned} g(f(x)) &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= -2x + 2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Composition of Functions

Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Composition of Functions

Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

[Show/Hide Solution]

Solution

The inside expression in the composition is $s(3)$. Because the input to the *s*-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Composition of Functions

There is yet another way to construct a new function from two functions. Figure 2.98 shows what happens when you apply one function $g(x) = x^2 - 1$ to an input (domain) value, x, and then apply a second function $f(x) = \sqrt{x}$ to the output value, $g(x)$, from the first function. In this case, the output from the first function becomes the input for the second function.

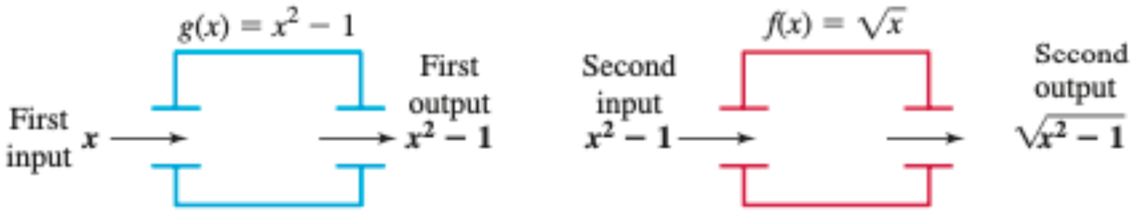

Figure 2.98 ▸ Composition of functions

Composition of Functions

If f and g are two functions, the composition of the function f with the function g is written as $f \circ g$ and is defined by the equation

$$(f \circ g)(x) = f(g(x)).$$

We read $(f \circ g)(x)$ as "f composed with g of x" and $f(g(x))$ as "f of g of x." The domain of $f \circ g$ consists of those values x in the domain of g for which $g(x)$ is in the domain of f.

Figure 2.99 may help clarify the definition of $f \circ g$.

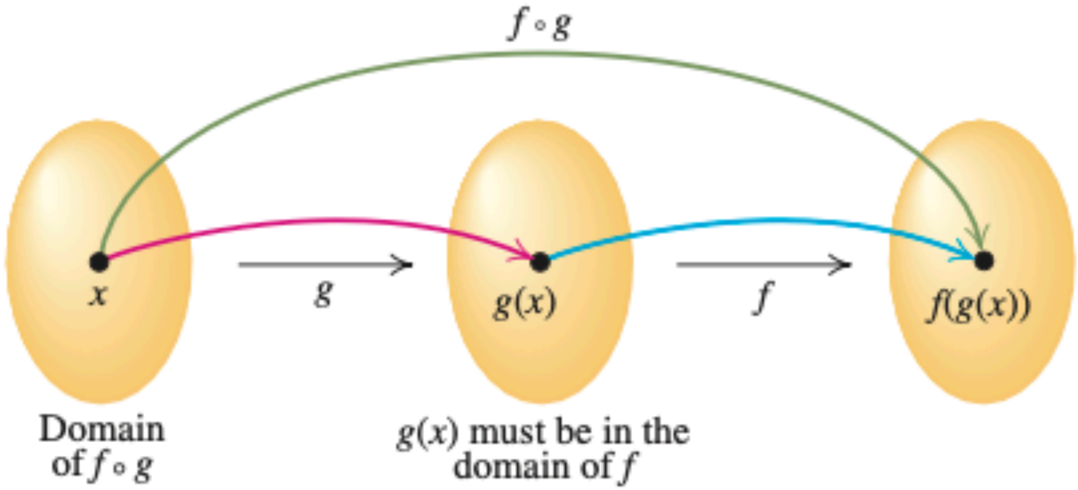

Figure 2.99 ▸ Composition of the function f with g

Evaluating $(f \circ g)(x)$ By definition,

inner function

$$(f \circ g)(x) = f(g(x)).$$

outer function

Thus, to evaluate $(f \circ g)(x)$, we must

(i) evaluate the inner function g at x.
(ii) use the result $g(x)$ as the input for the outer function f.

EXAMPLE 3 Evaluating a Composite Function

Let $f(x) = x^3$ and $g(x) = x + 1$. Find each composite function value.

a. $(f \circ g)(1)$ **b.** $(g \circ f)(1)$ **c.** $(f \circ f)(-1)$

EXAMPLE 3 Evaluating a Composite Function

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a. $(f \circ g)(1)$ **b.** $(g \circ f)(1)$ **c.** $(f \circ f)(-1)$

SIDE NOTE

When computing $(f \circ g)(x) = f(g(x))$, you must replace every x in $f(x)$ with $g(x)$.

Solution

a. $(f \circ g)(1) = f(g(1))$ Definition of $f \circ g$

$= [g(1)]^3$ Replace x with $g(1)$ in $f(x) = x^3$.

$= 2^3 = 8$ $g(1) = 1 + 1 = 2$; simplify.

SIDE NOTE

When computing $(f \circ g)(x) = f(g(x))$, you must replace every x in $f(x)$ with $g(x)$.

EXAMPLE 3 Evaluating a Composite Function

Let $f(x) = x^3$ and $g(x) = x + 1$. Find each composite function value.

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$= 2^3 = 8$ $g(1) = 1 + 1 = 2$; simplify.

b. $(g \circ f)(1) = g(f(1))$ Definition of $g \circ f$

$= f(1) + 1$ Replace x with $f(1)$ in $g(x) = x + 1$.

$= 1 + 1 = 2$ $f(1) = 1^3 + 1$; simplify.

SIDE NOTE

When computing $(f \circ g)(x) = f(g(x))$, you must replace every x in $f(x)$ with $g(x)$.

EXAMPLE 3 Evaluating a Composite Function

Let $f(x) = x^3$ and $g(x) = x + 1$. Find each composite function value.

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a. $(f \circ g)(1) = f(g(1))$ Definition of $f \circ g$

$= [g(1)]^3$ Replace x with $g(1)$ in $f(x) = x^3$.

$= 2^3 = 8$ $g(1) = 1 + 1 = 2$; simplify.

b. $(g \circ f)(1) = g(f(1))$ Definition of $g \circ f$

$= f(1) + 1$ Replace x with $f(1)$ in $g(x) = x + 1$.

$= 1 + 1 = 2$ $f(1) = 1^3 + 1$; simplify.

c. $(f \circ f)(-1) = f(f(-1))$ Definition of $f \circ f$

$= [f(-1)]^3$ Replace x with $f(-1)$ in $f(x) = x^3$.

$= (-1)^3 = -1$ $f(-1) = (-1)^3 = -1$; simplify.

Composition of Functions

1.5 Transformation of Functions

- Recall that velocity is a vector—it has both **magnitude** and **direction**. This means that a change in velocity can be a **change in magnitude** (or speed), but it can also be a **change in direction**. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

1.5 Tr

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f\left(x\right)$ is shifted vertically k units. See Figure 2 for an example.

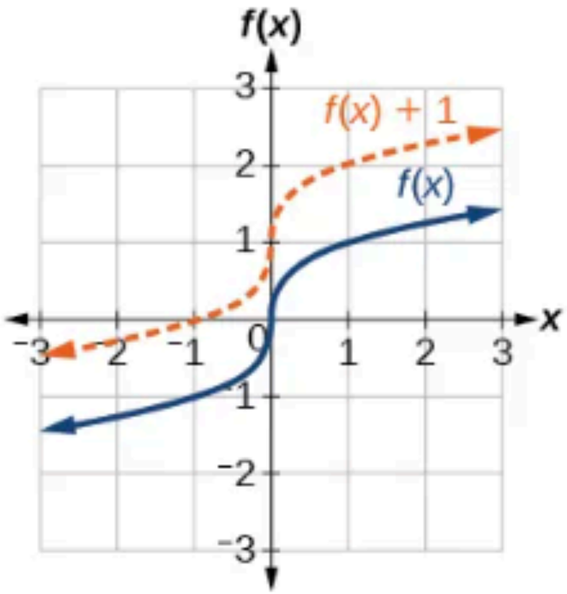


Figure 2 Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f\left(x\right)$. Therefore, $f\left(x\right) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y- value increases or decreases depending on the value of k. The result is a shift upward or downward.

EXAMPLE 1 Graphing Vertical Shifts

Let $f(x) = |x|$, $g(x) = |x| + 2$, and $h(x) = |x| - 3$. Sketch the graphs of these functions on the same coordinate plane. Describe how the graphs of g and h relate to the graph of f.

Solution

Make a table of values and graph the equations $y = f(x)$, $y = g(x)$, and $y = h(x)$.

TABLE 2.14

x	$y = \|x\|$	$g(x) = \|x\| + 2$	$h(x) = \|x\| - 3$
−5	5	7	2
−3	3	5	0
−1	1	3	−2
0	0	2	−3
1	1	3	−2
3	3	5	0
5	5	7	2

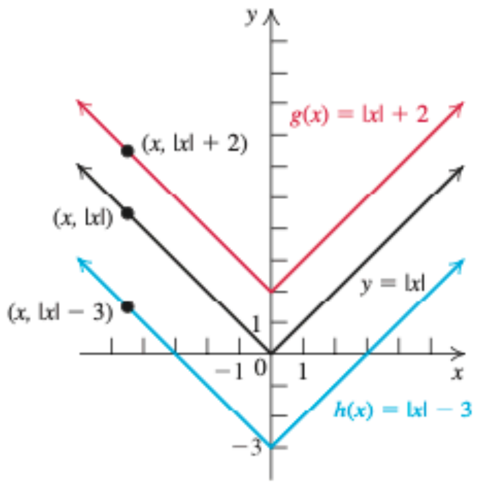


Figure 2.78 ▸ Vertical shifts of $y = |x|$

Notice that in Table 2.14 and Figure 2.78, each point $(x, |x|)$ on the graph of f has a corresponding point $(x, |x| + 2)$ on the graph of g and $(x, |x| - 3)$ on the graph of h. So the graph of $g(x) = |x| + 2$ is the graph of $y = |x|$ shifted 2 units up, and the graph of $y = |x| - 3$ is the graph of $y = |x|$ shifted 3 units down.

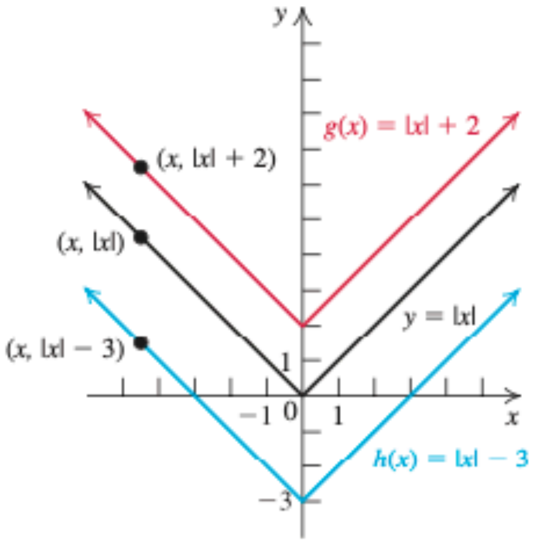


Figure 2.78 ▸ Vertical shifts of $y = |x|$

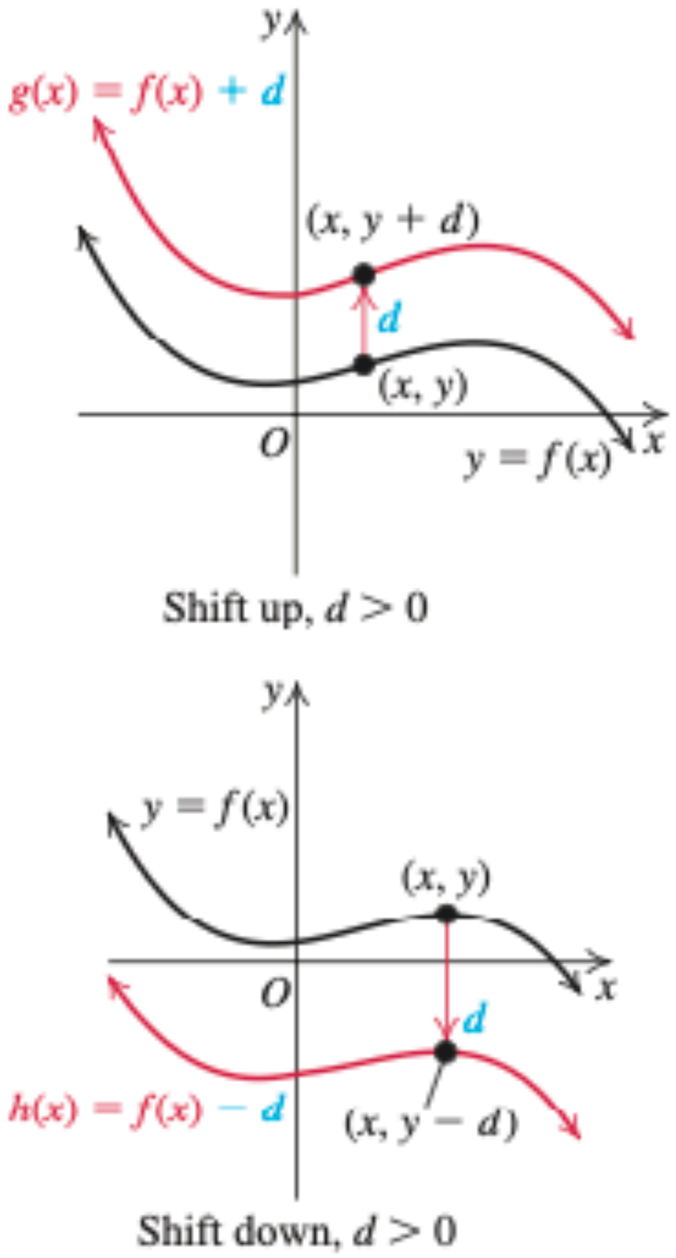


Figure 2.79 ► Vertical shifts

Practice Problem 1. Let

$$f(x) = x^3,\ g(x) = x^3 + 1, \text{ and } h(x) = x^3 - 2.$$

Sketch the graphs of all three functions on the same coordinate plane. Describe how the graphs of g and h relate to the graph of f. ■

Example 1 illustrates the concept of the vertical (up or down) shift of a graph.

Vertical Shift

Let $d > 0$. The graph of $g(x) = f(x) + d$ is the graph of $y = f(x)$ shifted d units *up*, and the graph of $h(x) = f(x) - d$ is the graph of $y = f(x)$ shifted d units *down*. See Figure 2.79. If (x, y) is a point on the graph of $y = f(x)$, then the corresponding point $(x, y + d)$ is on the graph of $g(x) = f(x) + d$ and the point $(x, y - d)$ is on the graph of $h(x) = f(x) - d$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 5.

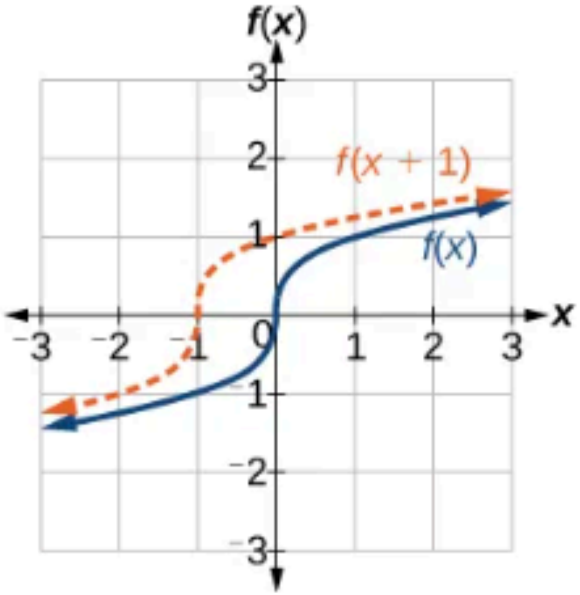


Figure 5 Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $(x+1)$ means $h = -1$ which shifts the graph to the left, that is, towards *negative* values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

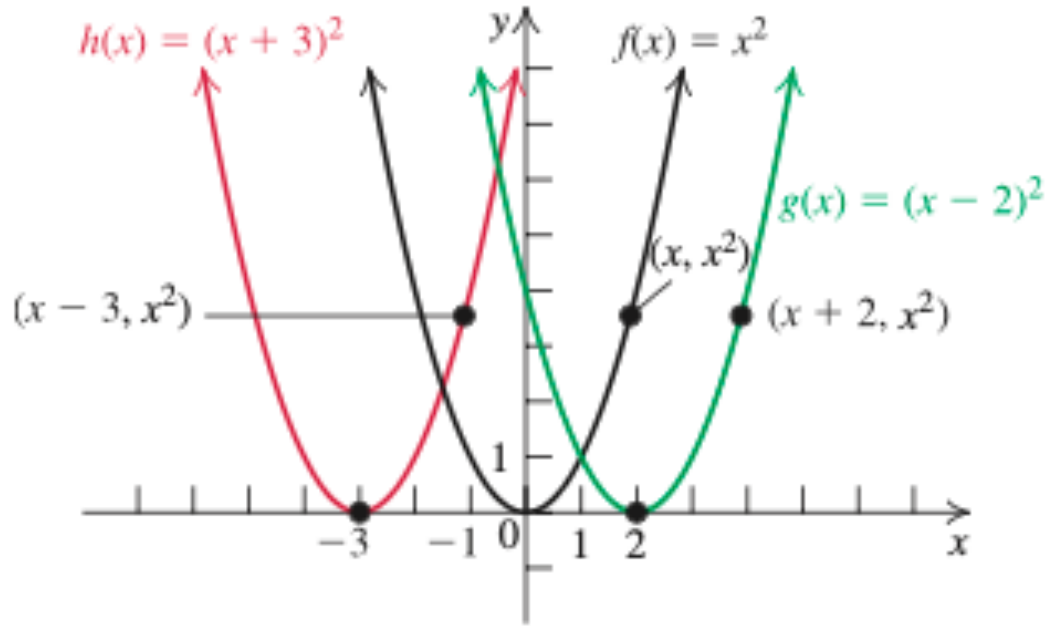


Figure 2.80 ▸ Horizontal shifts

EXAMPLE 2 Writing Functions for Horizontal Shifts

Let $f(x) = x^2$, $g(x) = (x - 2)^2$, and $h(x) = (x + 3)^2$. A table of values for f, g, and h is given in Table 2.15. The three functions f, g, and h are graphed on the same coordinate plane in Figure 2.80. Describe how the graphs of g and h relate to the graph of f.

TABLE 2.15

x	$f(x) = x^2$	$g(x) = (x - 2)^2$
−4	**16**	36
−3	**9**	25
−2	**4**	16
−1	**1**	9
0	**0**	4
1	**1**	1
2	4	0
3	9	1
4	16	4

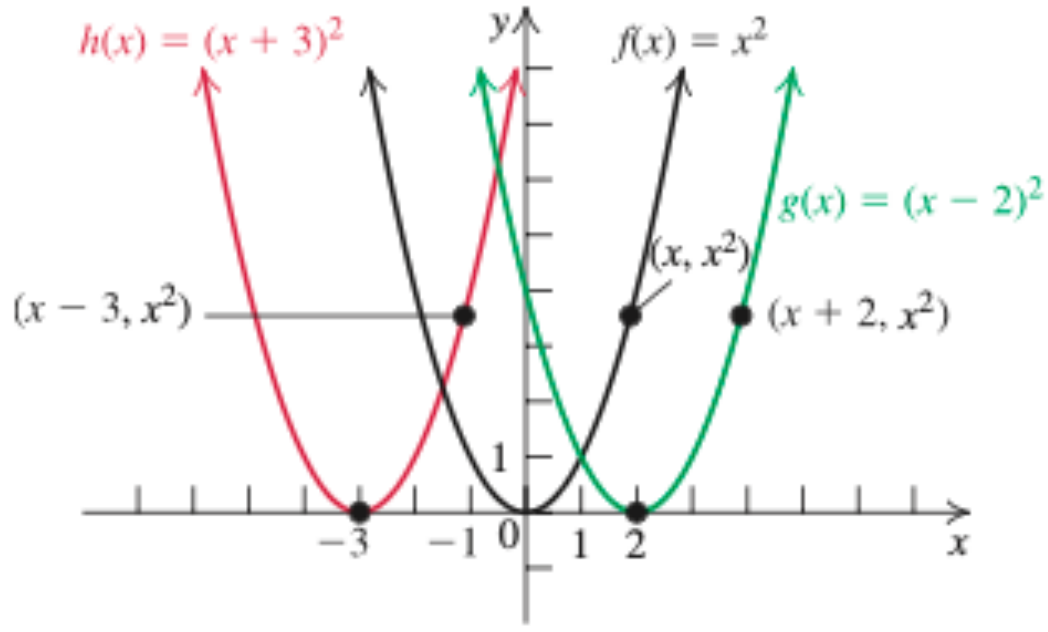
(a)

x	$f(x) = x^2$	$h(x) = (x + 3)^2$
−4	16	1
−3	9	0
−2	4	1
−1	**1**	4
0	**0**	9
1	**1**	16
2	**4**	25
3	**9**	36
4	**16**	49

(b)

First, notice that all three functions are squaring functions.

a. Each point (x, x^2) on the graph of f has a corresponding point $(x + 2, x^2)$ on the graph of g because $g(x + 2) = (x + 2 - 2)^2 = x^2$. So the graph of $g(x) = (x - 2)^2$ is just the graph of $f(x) = x^2$ shifted 2 units to the *right*. Noticing that $f(0) = \mathbf{0}$ and $g(2) = \mathbf{0}$ will help you remember which way to shift the graph. Table 2.15(a) and Figure 2.80 illustrate these ideas.



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−1	**1**	9
0	**0**	4
1	**1**	1
2	4	0
3	9	1
4	16	4

(b)

x	$f(x) = x^2$	$h(x) = (x + 3)^2$
−4	16	1
−3	9	0
−2	4	1
−1	**1**	4
0	**0**	9
1	**1**	16
2	**4**	25
3	**9**	36
4	**16**	49

First, notice that all three functions are squaring functions.

a. Each point (x, x^2) on the graph of f has a corresponding point $(x + 2, x^2)$ on the graph of g because $g(x + 2) = (x + 2 - 2)^2 = x^2$. So the graph of $g(x) = (x - 2)^2$ is just the graph of $f(x) = x^2$ shifted 2 units to the *right*. Noticing that $f(0) = \mathbf{0}$ and $g(2) = \mathbf{0}$ will help you remember which way to shift the graph. Table 2.15(a) and Figure 2.80 illustrate these ideas.

b. Each point (x, x^2) on the graph of f has a corresponding point $(x - 3, x^2)$ on the graph of h because $h(x - 3) = (x - 3 + 3)^2 = x^2$. So the graph of $h(x) = (x + 3)^2$ is just the graph of $f(x) = x^2$ shifted 3 units to the *left*. Noticing that $f(0) = \mathbf{0}$ and $h(-3) = \mathbf{0}$ will help you remember which way to shift the graph. Table 2.15(b) and Figure 2.80 confirm these considerations.

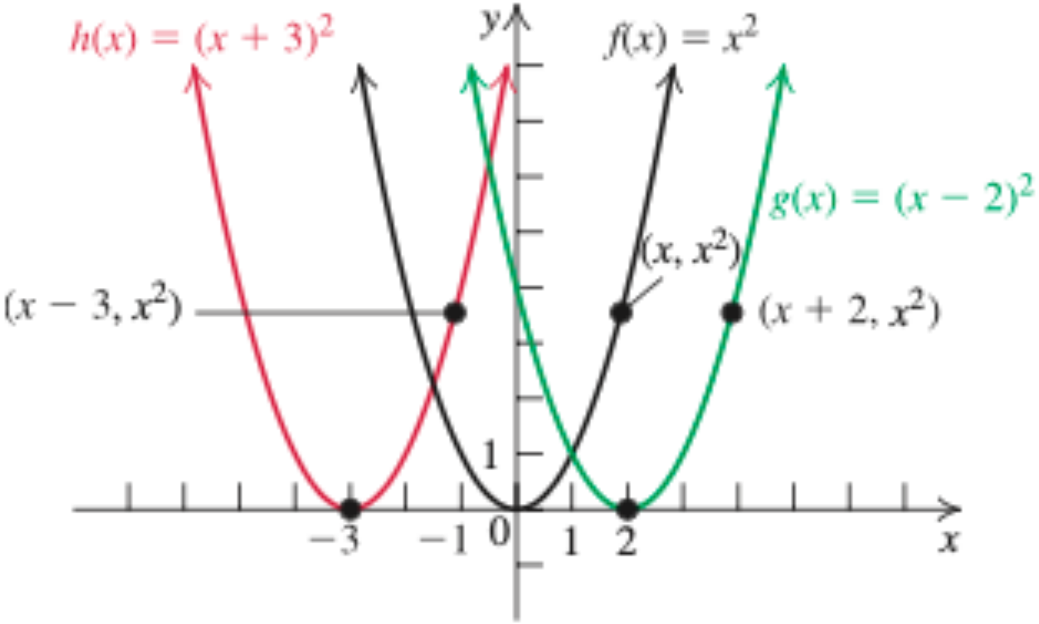

Figure 2.80 ▸ Horizontal shifts

EXAMPLE 2 Writing Functions for Horizontal Shifts

Let $f(x) = x^2$, $g(x) = (x-2)^2$, and $h(x) = (x+3)^2$. A table of values for f, g, and h is given in Table 2.15. The three functions f, g, and h are graphed on the same coordinate plane in Figure 2.80. Describe how the graphs of g and h relate to the graph of f.

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2	4	0
3	9	1
4	16	4

(a)

x	$f(x) = x^2$	$h(x) = (x+3)^2$
−4	16	1
−3	9	0
−2	4	1
−1	**1**	4
0	**0**	9
1	**1**	16
2	**4**	25
3	**9**	36
4	**16**	49

(b)

Horizontal Shifts

Let $c > 0$. The graph of $y = f(x-c)$ is the graph of $y = f(x)$ shifted c units to the right. The graph of $y = f(x+c)$ is the graph of $y = f(x)$ shifted c units to the left. See Figure 2.81. If (x, y) is a point on the graph of f, then the corresponding point $(x+c, y)$ is on the graph of $y = f(x-c)$ and the point $(x-c, y)$ is on the graph of $y = f(x+c)$.

Horizontal Shifts

Let $c > 0$. The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units to the right. The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted c units to the left. See Figure 2.81. If (x, y) is a point on the graph of f, then the corresponding point $(x + c, y)$ is on the graph of $y = f(x - c)$ and the point $(x - c, y)$ is on the graph of $y = f(x + c)$.

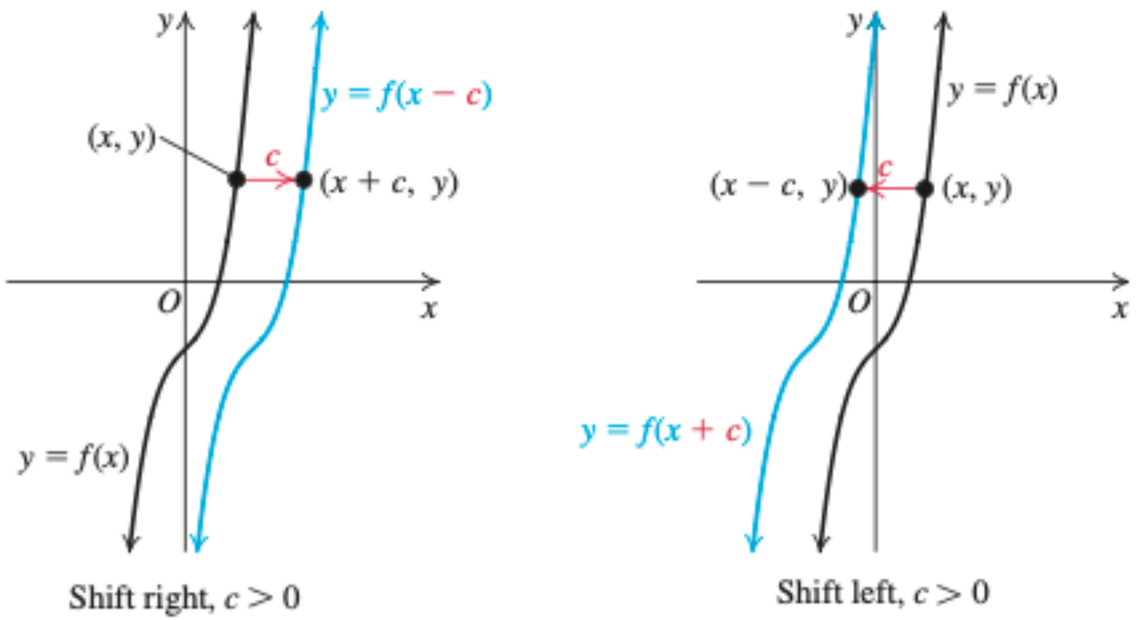


Figure 2.81 ▸ Horizontal shifts

Suggested practice problems.

Practice Problem 2. Let

$$f(x) = x^3,\ g(x) = (x - 1)^3, \text{ and } h(x) = (x + 2)^3.$$

Sketch the graphs of all three functions on the same coordinate plane. Describe how the graphs of g and h relate to the graph of f. ■

PROCEDURE IN ACTION

EXAMPLE 3 Graphing Combined Vertical and Horizontal Shifts

Objective	Example
Sketch the graph of $g(x) = f(x \pm c) \pm d$, *where* f *is a function whose graph is known,* $c > 0$, $d > 0$.	*Sketch the graph of* $g(x) = \sqrt{x + 2} - 3$.

PROCEDURE IN ACTION

EXAMPLE 3 Graphing Combined Vertical and Horizontal Shifts

Objective	Example
Sketch the graph of $g(x) = f(x \pm c) \pm d$, *where* f *is a function whose graph is known,* $c > 0$, $d > 0$.	*Sketch the graph of* $g(x) = \sqrt{x+2} - 3$.
Step 1 ▸ Identify and graph the known function f.	**1.** Choose $f(x) = \sqrt{x}$. Domain: $[0, \infty)$; range: $[0, \infty)$. The graph of $y = \sqrt{x}$ is shown in Step 4.
Step 2 ▸ Identify the *positive* constants c and d.	**2.** $g(x) = \sqrt{x + 2} - 3$, so $c = 2$ and $d = 3$.
Step 3 ▸ **(i)** Graph $y = f(x - c)$ by shifting the graph of f horizontally c units to the right. Add c to the x-coordinate of each point. **(ii)** Graph $y = f(x + c)$ by shifting the graph of f horizontally c units to the left. Subtract c from the x-coordinate of each point.	**3.** Because $c = 2 > 0$, the graph of $y = \sqrt{x+2}$ is the graph of f shifted horizontally 2 units to the left. See the blue graph in Step 4.

PROCEDURE IN ACTION

EXAMPLE 3 Graphing Combined Vertical and Horizontal Shifts

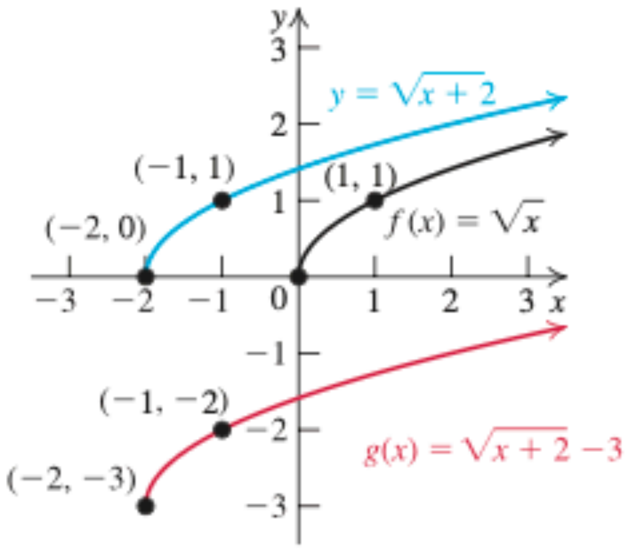
Objective	Example
Sketch the graph of $g(x) = f(x \pm c) \pm d$, *where f is a*	*Sketch the graph of* $g(x) = \sqrt{x+2} - 3$.

Step 4 ▸

(i) Graph $y = f(x \pm c) + d$ by shifting the graph of $y = f(x \pm c)$ vertically up d units. Add d to the y-coordinate of each point.

(ii) Graph $y = f(x \pm c) - d$ by shifting the graph of $y = f(x \pm c)$ vertically down d units. Subtract d from the y-coordinate of each point.

4. Graph $y = \sqrt{x+2} - 3$ by shifting the graph of $y = \sqrt{x+2}$ down 3 units. Domain: $[-2, \infty)$; range: $[-3, \infty)$. See the red graph below.



Horizontal and vertical shifts

Practice Problem 3. Sketch the graph of

$$f(x) = \sqrt{x-2} + 3.$$

■

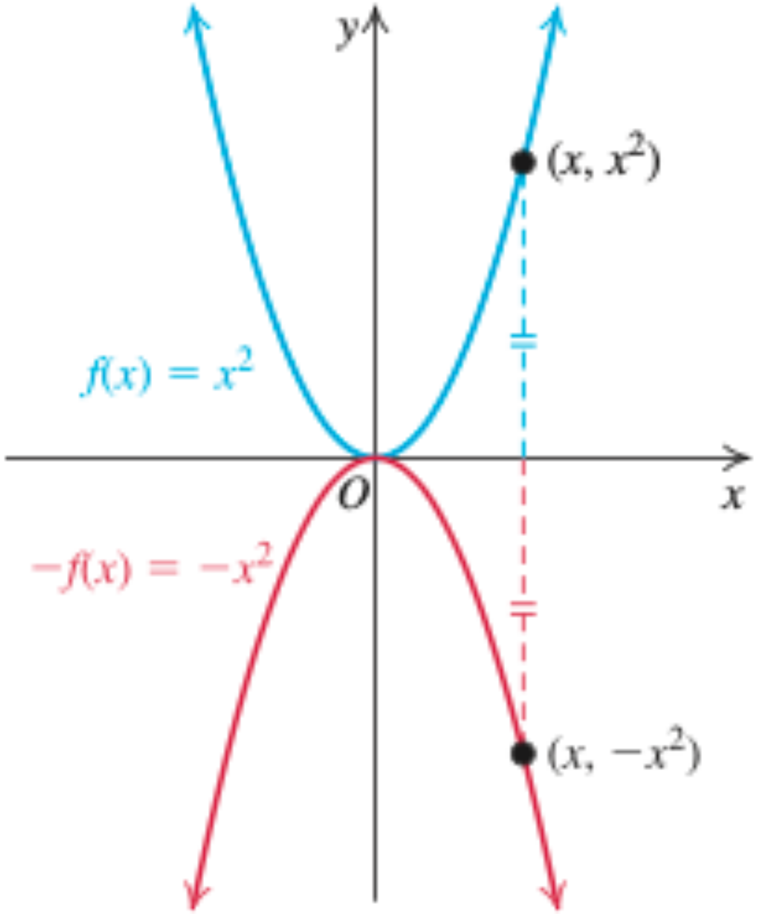


Figure 2.82 ▸ Reflection about the x-axis

Reflections

We now consider rigid transformations that reflect a graph about a coordinate axis.

Comparing the Graphs of $y = f(x)$ and $y = -f(x)$ Consider the graph of $f(x) = x^2$, shown in Figure 2.82. Table 2.16 gives some values for $f(x)$ and $g(x) = -f(x)$. Note that the y-coordinate of each point in the graph of $g(x) = -f(x)$ is the opposite of the y-coordinate of the corresponding point on the graph of $f(x)$. So the graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ about the x-axis. This means that the points (x, x^2) and $(x, -x^2)$ are the same distance from, but on opposite sides of, the x-axis. See Figure 2.82.

TABLE 2.16

x	$f(x) = x^2$	$g(x) = -f(x) = -x^2$
−3	9	−9
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4
3	9	−9

Reflection about the x-axis

The graph of $g(x) = -f(x)$ is a reflection of the graph of $y = f(x)$ about the x-axis. If a point (x, y) is on the graph of f, then the point $(x, -y)$ is on the graph of g. See Figure 2.83.

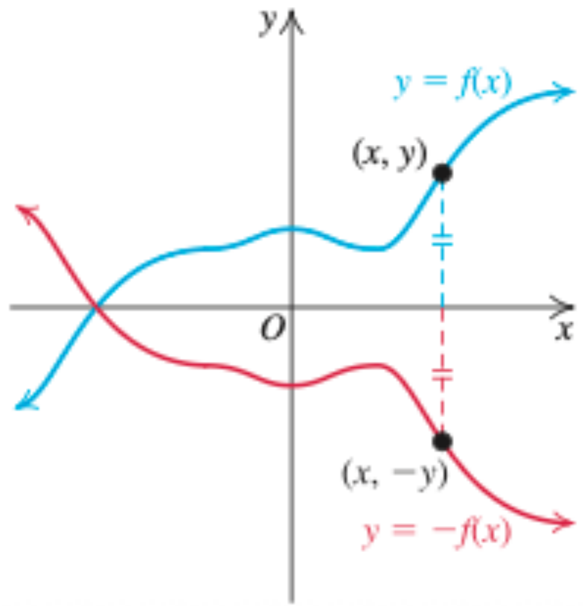


Figure 2.83 ▸ Reflection about the x-axis

Reflections

We now consider rigid transformations that reflect a graph about a coordinate axis.

Comparing the Graphs of $y = f(x)$ and $y = -f(x)$ Consider the graph of $f(x) = x^2$, shown in Figure 2.82. Table 2.16 gives some values for $f(x)$ and $g(x) = -f(x)$. Note that the y-coordinate of each point in the graph of $g(x) = -f(x)$ is the opposite of the y-coordinate of the corresponding point on the graph of $f(x)$. So the graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ about the x-axis. This means that the points (x, x^2) and $(x, -x^2)$ are the same distance from, but on opposite sides of, the x-axis. See Figure 2.82.

TABLE 2.16

x	$f(x) = x^2$	$g(x) = -f(x) = -x^2$
−3	9	−9
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4
3	9	−9

Reflection about the x-axis

The graph of $g(x) = -f(x)$ is a reflection of the graph of $y = f(x)$ about the x-axis. If a point (x, y) is on the graph of f, then the point $(x, -y)$ is on the graph of g. See Figure 2.83.

TABLE 2.17

x	$f(x) = \sqrt{x}$	$-x$	$g(x) = \sqrt{-x}$
-4	Undefined	4	2
-1	Undefined	1	1
0	0	0	0
1	1	-1	Undefined
4	2	-4	Undefined

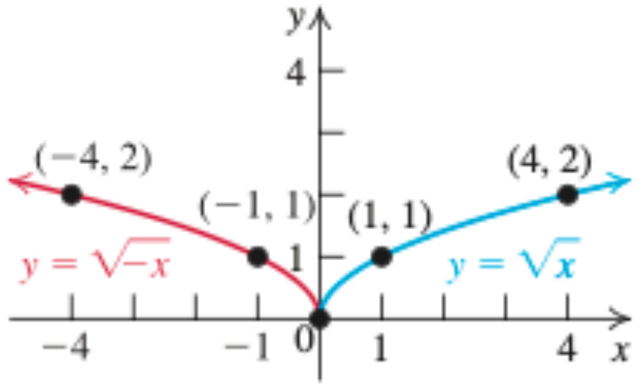


FIGURE 2.84 ▸ Graphing $y = \sqrt{-x}$

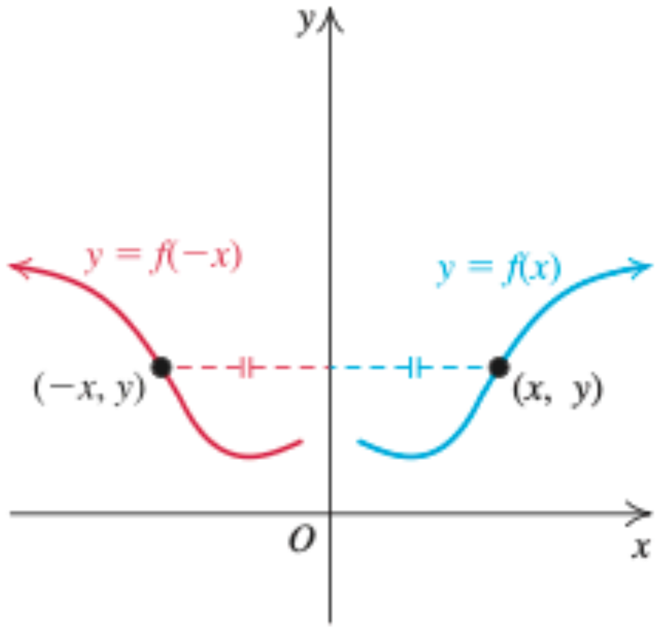


Figure 2.85 ▸ Reflection about the y-axis

Reflection about the y-axis

The graph of $g(x) = f(-x)$ is a reflection of the graph of $y = f(x)$ about the y-axis. If a point (x, y) is on the graph of f, then the point $(-x, y)$ is on the graph of g. See Figure 2.85.

EXAMPLE 4 Combining Transformations

Explain how the graph of $y = -|x - 2| + 3$ can be obtained from the graph of $y = |x|$.

EXAMPLE 4 Combining Transformations

Explain how the graph of $y = -|x - 2| + 3$ can be obtained from the graph of $y = |x|$.

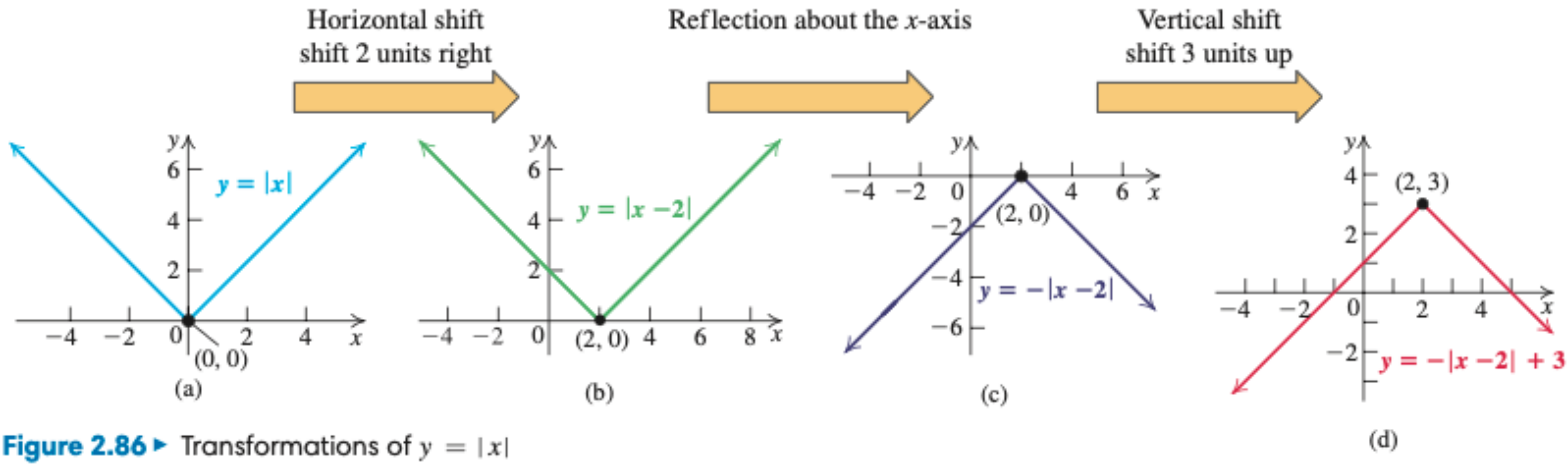


Figure 2.86 ▸ Transformations of $y = |x|$

Suggested practice problems

Practice Problem 4. Explain how the graph of $y = -(x-1)^2 + 2$ can be obtained from the graph of $y = x^2$. ■

Vertical Stretching or Compressing

The graph of $g(x) = af(x)$ is obtained from the graph of $y = f(x)$ by multiplying the y-coordinate of each point on the graph of $y = f(x)$ by a and leaving the x-coordinate unchanged. The result (see Figure 2.89) is as follows:

1. A **vertical stretch** away from the x-axis if $a > 1$
2. A **vertical compression** toward the x-axis if $0 < a < 1$

If $a < 0$, first graph $y = |a| f(x)$ by stretching or compressing the graph of $y = f(x)$ vertically. Then reflect the resulting graph about the x-axis.

So if (x, y) is a point on the graph of f, then the point (x, ay) is on the graph of g.

With both vertical stretching and compressing, the x-intercepts remain the same but the y-intercept may change.

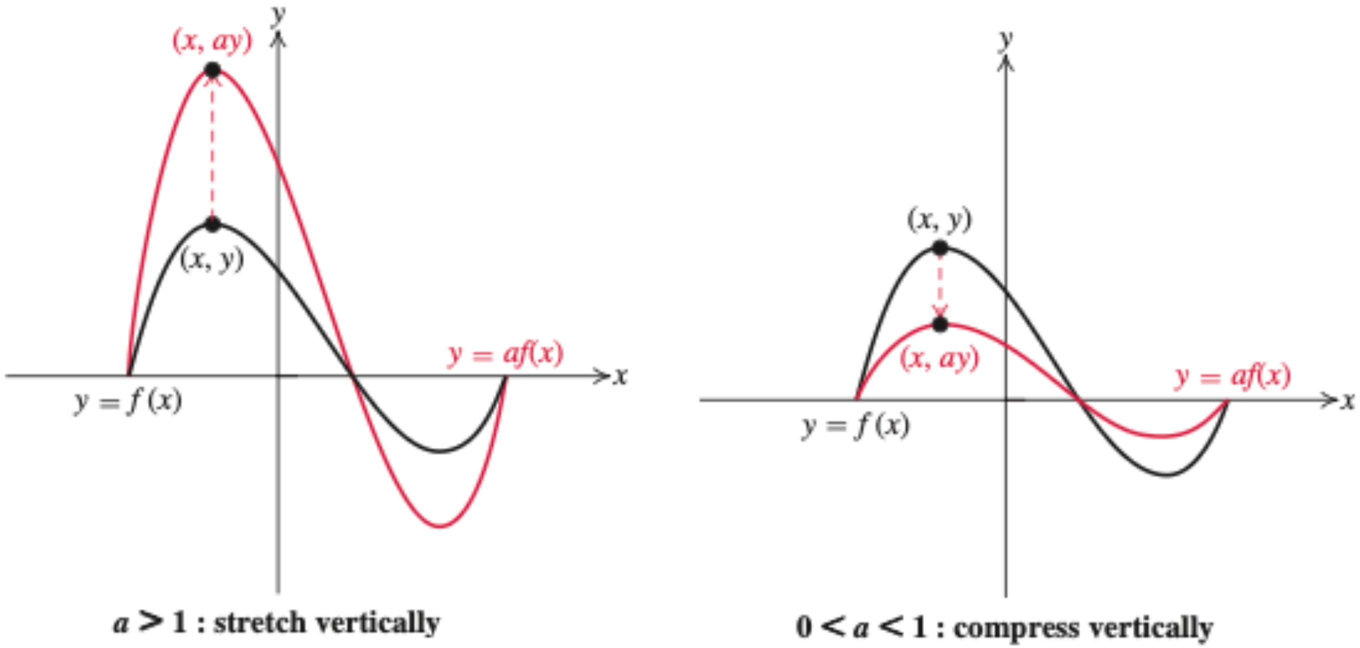


Figure 2.89 ▸ Vertical stretch or compression

Stretching or Compressing

We now look at transformations that distort the shape of a graph, called **nonrigid transformations**. We consider the relationship of the graphs of $y = af(x)$ and $y = f(bx)$ to the graph of $y = f(x)$.

Comparing the Graphs of $y = f(x)$ and $y = af(x)$

EXAMPLE 6 **Stretching or Compressing a Graph Vertically**

Let $f(x) = |x|$, $g(x) = 2|x|$, and $h(x) = \frac{1}{2}|x|$. Sketch the graphs of f, g, and h on the same coordinate plane and describe how the graphs of g and h are related to the graph of f.

Solution

The graphs of $y = |x|$, $y = 2|x|$, and $y = \frac{1}{2}|x|$ are sketched in Figure 2.88. Table 2.18 gives some typical function values.

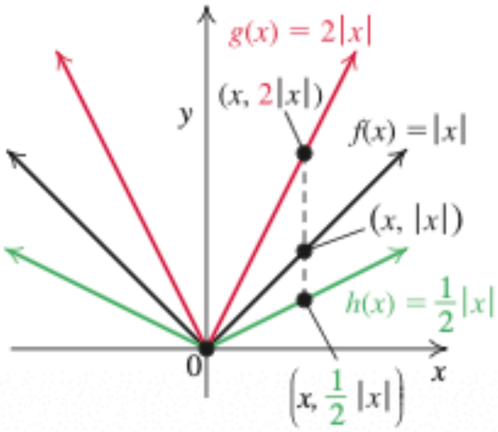


Figure 2.88 ▸ Vertical stretch and compression

TABLE 2.18

x	$f(x) = \lvert x \rvert$	$g(x) = 2\lvert x \rvert$	$h(x) = \frac{1}{2}\lvert x \rvert$
−2	2	4	1
−1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	2	4	1

Vertical Stretching or Compressing

The graph of $g(x) = af(x)$ is obtained from the graph of $y = f(x)$ by multiplying the y-coordinate of each point on the graph of $y = f(x)$ by a and leaving the x-coordinate unchanged. The result (see Figure 2.89) is as follows:

1. A **vertical stretch** away from the x-axis if $a > 1$
2. A **vertical compression** toward the x-axis if $0 < a < 1$

If $a < 0$, first graph $y = |a|\, f(x)$ by stretching or compressing the graph of $y = f(x)$ vertically. Then reflect the resulting graph about the x-axis.

So if (x, y) is a point on the graph of f, then the point (x, ay) is on the graph of g.
With both vertical stretching and compressing, the x-intercepts remain the same but the y-intercept may change.

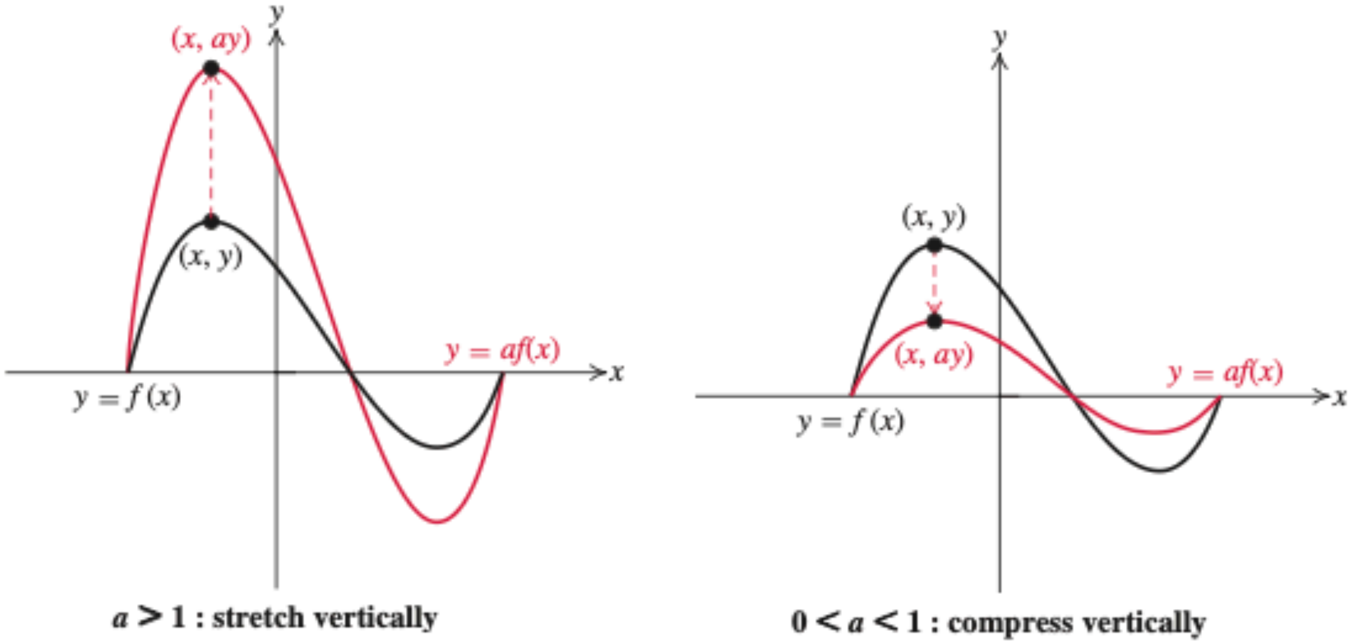
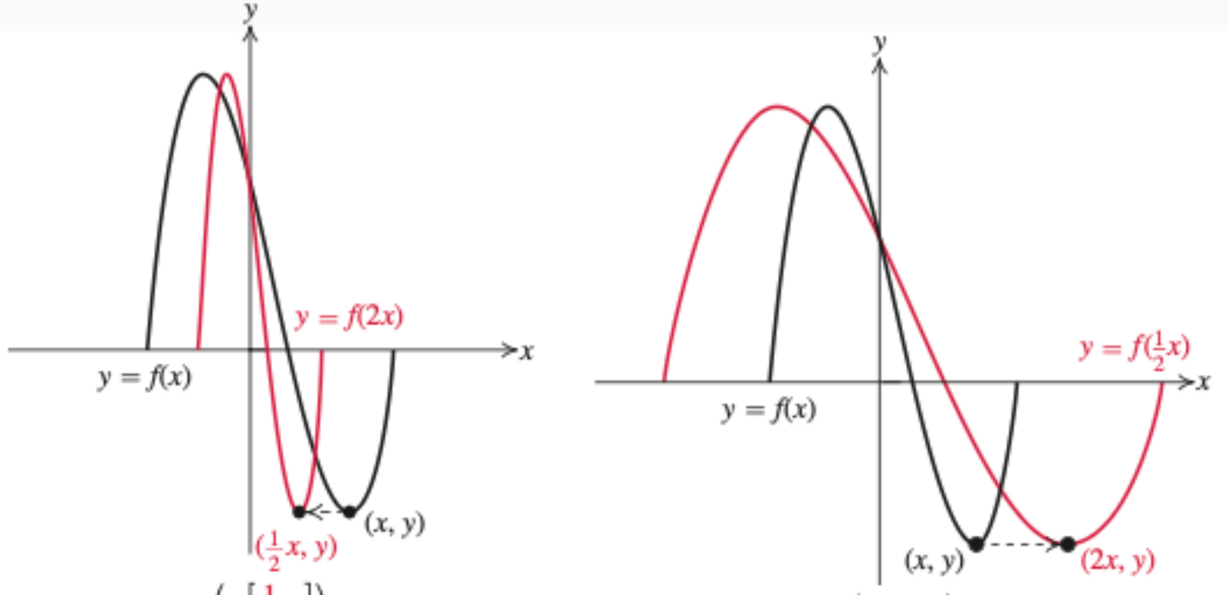


Figure 2.89 ▸ Vertical stretch or compression



(a) $f\left(2\left[\frac{1}{2}x\right]\right) = f(x)$ (b) $f\left(\frac{1}{2}[2x]\right) = f(x)$

Figure 2.90 ▸ Horizontal stretch or compression

Horizontal Stretching or Compressing

Let $b > 0$. The graph of $g(x) = f(bx)$ is obtained from the graph of $y = f(x)$ by multiplying the x-coordinate of each point on the graph of $y = f(x)$ by $\frac{1}{b}$ and leaving the y-coordinate unchanged. The result (see Figure 2.91) is as follows:

1. A **horizontal stretch** away from the y-axis if $0 < b < 1$
2. A **horizontal compression** toward the y-axis if $b > 1$

If $b < 0$, first graph $f(|b|x)$ by stretching or compressing the graph of $y = f(x)$ horizontally. Then reflect the graph of $y = f(|b|x)$ about the y-axis.

If (x, y) is a point on the graph of $y = f(x)$, then the point $\left(\frac{1}{b}x, y\right)$ is on the graph of $g(x) = f(bx)$.

With both horizontal stretching and compressing, the y-intercept remains the same but the x-intercepts may be affected.

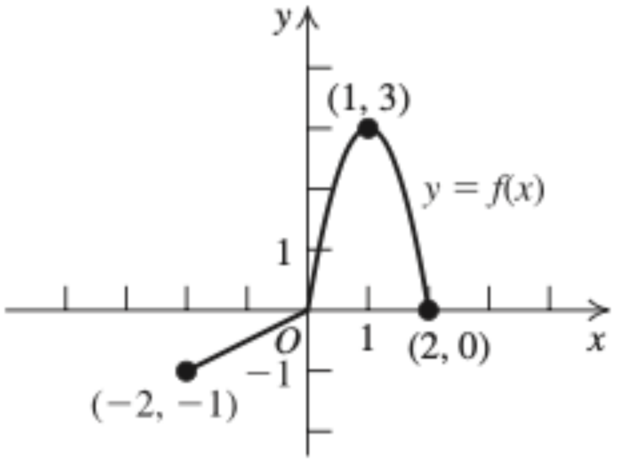


Figure 2.92

EXAMPLE 7 Stretching or Compressing a Function Horizontally

The graph of the function $y = f(x)$ whose equation is not given is shown in Figure 2.92. Sketch the following graphs.

d. $f\left(\frac{1}{2}x\right)$ **b.** $f(2x)$ **c.** $f(-2x)$

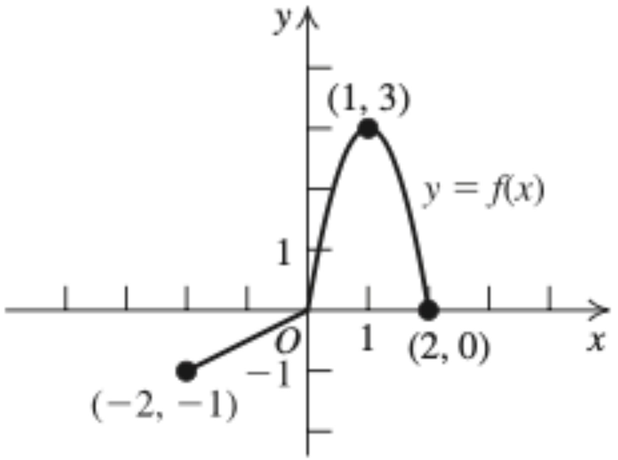


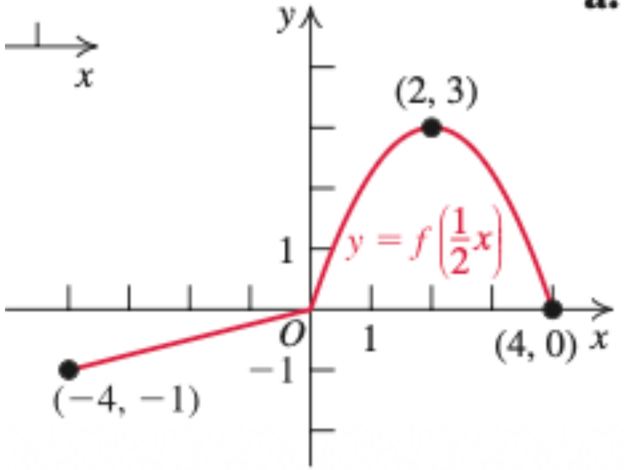
Figure 2.92

EXAMPLE 7 Stretching or Compressing a Function Horizontally

The graph of the function $y = f(x)$ whose equation is not given is shown in Figure 2.92. Sketch the following graphs.

d. $f\left(\frac{1}{2}x\right)$ **b.** $f(2x)$ **c.** $f(-2x)$

Note that the domain of f is $[-2,\ 2]$ and its range is $[-1,\ 3]$.



Domain: [−4, 4]; range: [−1, 3]

(a)

a. To graph $y = f\left(\frac{1}{2}x\right)$, we stretch the graph of $y = f(x)$ horizontally by a factor of 2. In other words, we transform each point $(x,\ y)$ in Figure 2.92 to the point $(2x,\ y)$. The x-intercept 2 changes to 4, but the y-intercept is the same. See Figure 2.93(a).

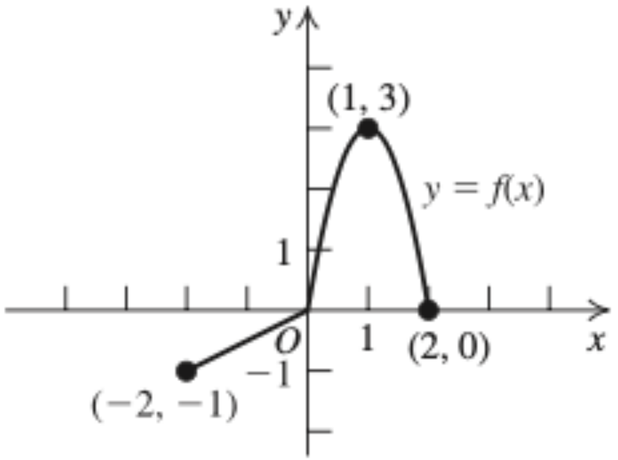


Figure 2.92

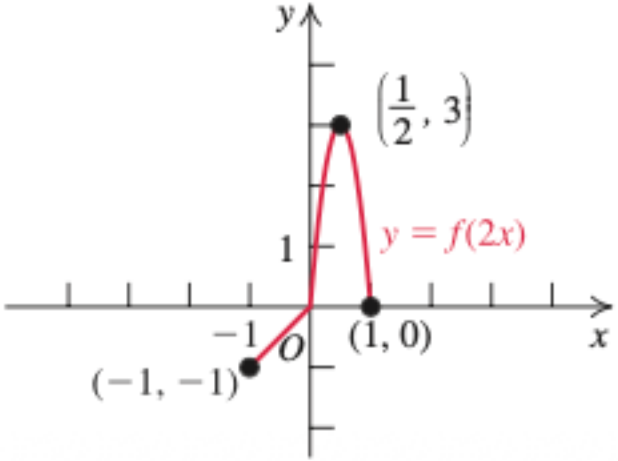
EXAMPLE 7 Stretching or Compressing a Function Horizontally

The graph of the function $y = f(x)$ whose equation is not given is shown in Figure 2.92. Sketch the following graphs.

d. $f\left(\frac{1}{2}x\right)$ **b.** $f(2x)$ **c.** $f(-2x)$

Note that the domain of f is $[-2, 2]$ and its range is $[-1, 3]$.

b. To graph $y = f(2x)$, we compress the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{2}$. Therefore, we transform each point (x, y) in Figure 2.92 to $\left(\frac{1}{2}x, y\right)$. The x-intercept 2 changes to 1, but the y-intercept is the same. See Figure 2.93(b).



Domain: [−1, 1]; range: [−1, 3]

(b)

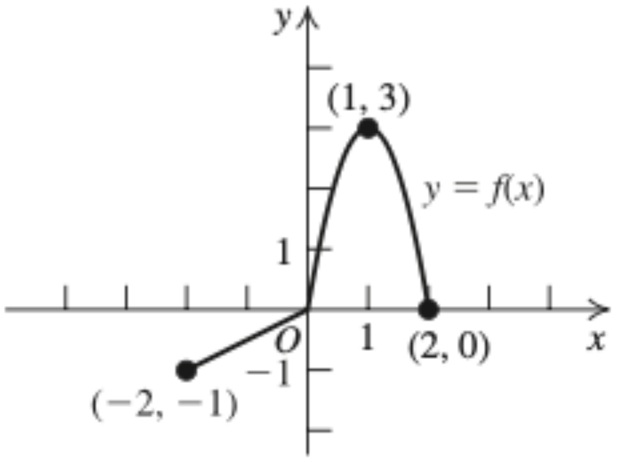


Figure 2.92

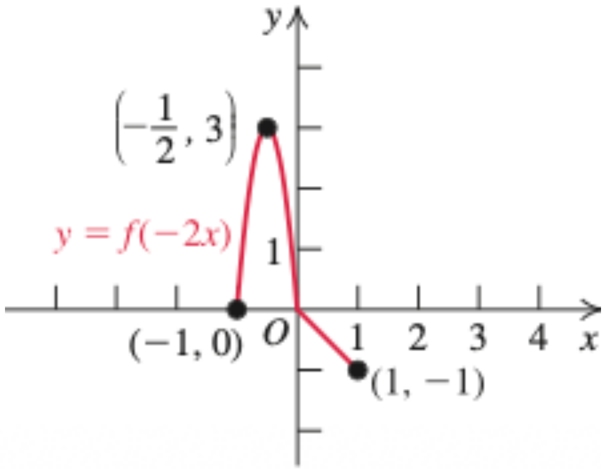
EXAMPLE 7 Stretching or Compressing a Function Horizontally

The graph of the function $y = f(x)$ whose equation is not given is shown in Figure 2.92. Sketch the following graphs.

d. $f\left(\frac{1}{2}x\right)$ **b.** $f(2x)$ **c.** $f(-2x)$

Note that the domain of f is $[-2, 2]$ and its range is $[-1, 3]$.

c. To graph $y = f(-2x)$, we reflect the graph of $y = f(2x)$ in Figure 2.93(b) about the y-axis. So we transform each point (x, y) in Figure 2.93(b) to the point $(-x, y)$. The x-intercept 1 changes to -1, but the y-intercept is the same. See Figure 2.93(c).



Domain: [−1, 1]; range: [−1, 3]

(c)

Suggested practice problem

Practice Problem 8. Sketch the graph of the function

$$f(x) = 3\sqrt{x+1} - 2.$$

SUMMARY OF MAIN FACTS

Summary of Transformations of $y = f(x)$

To graph	Draw the graph of f and	Make these changes to the equation $y = f(x)$	Change the graph point (x, y) to
Vertical shift (for $d > 0$)	*Shift the graph of* f		
$y = f(x) + d$	up d units.	Add d to $f(x)$.	$(x, y + d)$
$y = f(x) - d$	down d units.	Subtract d from $f(x)$.	$(x, y - d)$
Horizontal shift (for $c > 0$)	*Shift the graph of* f		
$y = f(x - c)$	to the right c units.	Replace x with $x - c$.	$(x + c, y)$
$y = f(x + c)$	to the left c units.	Replace x with $x + c$.	$(x - c, y)$

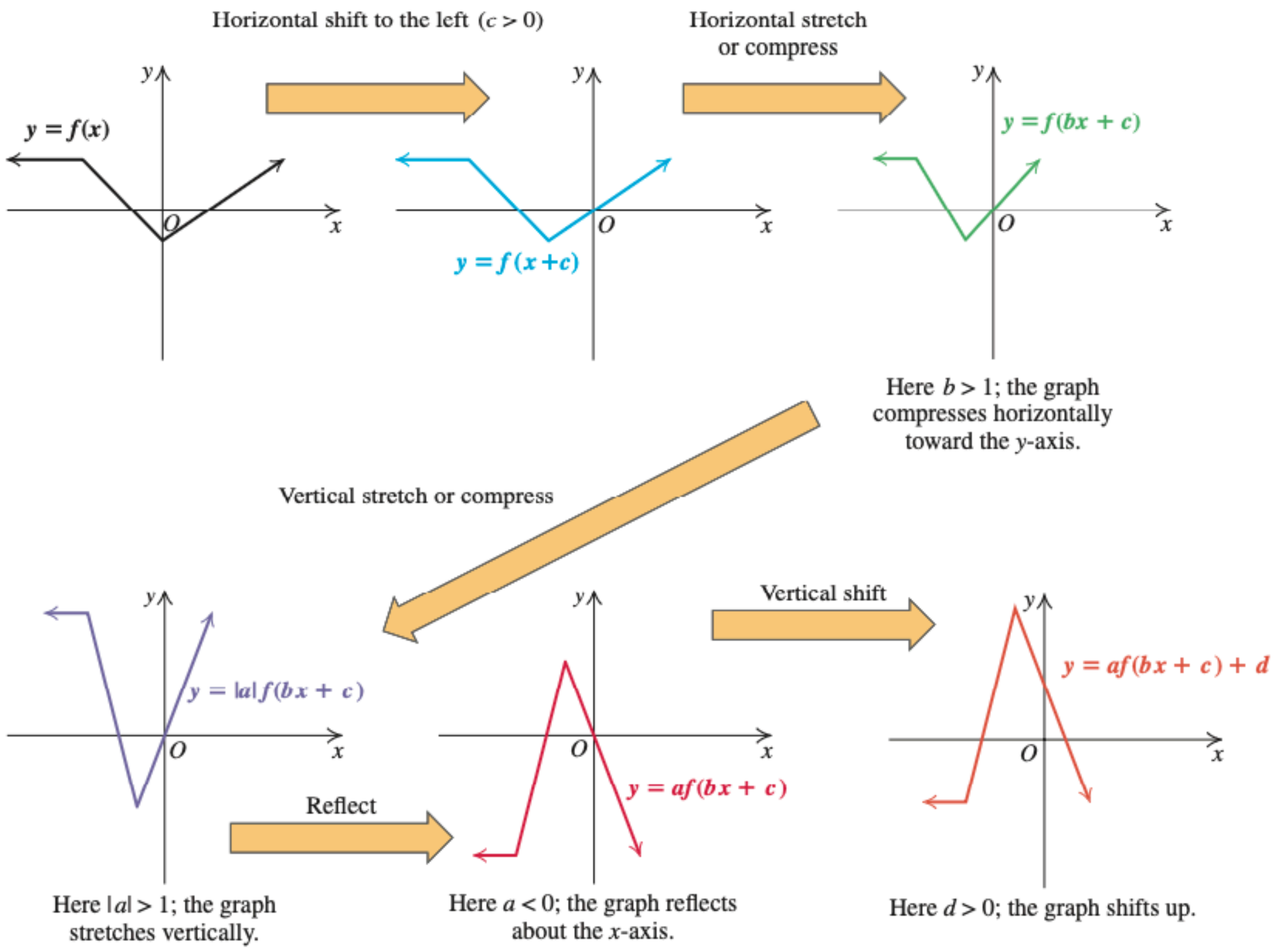
To graph	Draw the graph of f and	Make these changes to the equation $y = f(x)$	Change the graph point (x, y) to
Reflection about the x-axis			
$y = -f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.	$(x, -y)$
Reflection about the y-axis			
$y = f(-x)$	Reflect the graph of f about the y-axis.	Replace x with $-x$.	$(-x, y)$
Vertical stretching or compressing: $y = af(x)$	Multiply each y-coordinate of $y = f(x)$ by a. The graph of $y = f(x)$ is stretched vertically away from the x-axis if $a > 1$ and is compressed vertically toward the x-axis if $0 < a < 1$. If $a < 0$, sketch the graph of $y = \lvert a \rvert f(x)$, then reflect it about the x-axis.	Multiply $f(x)$ by a.	(x, ay)

Horizontal stretching or compressing: $y = f(bx)$	Multiply each x-coordinate of $y = f(x)$ by $\frac{1}{b}$. The graph of $y = f(x)$ is stretched away from the y-axis if $0 < b < 1$ and is compressed toward the y-axis if $b > 1$. If $b < 0$, sketch the graph of $y = f(\lvert b\rvert x)$, then reflect it about the y-axis.	Replace x with bx.	$\left(\frac{x}{b}, y\right)$

Summary for Combining Transformations in Sequence

To graph $y = af(bx + c) + d$, with $b > 0$ and $c > 0$

- Starting with the graph of $y = f(x)$, perform the indicated sequence of transformations. Here is one possible sequence:



Note: If $c < 0$, then the horizontal shift should be $|c|$ units to the right, if $b < 0$, graph $y = af(|b|x + c) + d$, but reflect this graph about the y-axis before completing the last step, the vertical shift, d.

Multiple Transformations in Sequence

When graphing requires more than one transformation of a basic function, it is helpful to perform transformations in the following order:

1. Horizontal shift **2. Stretch or Compression** **3. Reflection** **4. Vertical shift**

Below we present one possible order of applying transformations to the graph of $y = f(x)$. Here we assume the numbers a, b, c, and d are positive.

Graph	$\boldsymbol{y = \pm af(\pm bx \pm c) \pm d}$	
Case I	$y = af(bx \pm c) \pm d$	Positive leading coefficient, a
Case II	$y = -af(bx \pm c) \pm d$	Negative leading coefficient, $-a$
Case III	$y = af(-bx \pm c) \pm d$	Negative coefficient for x, $-b$
Case IV	$y = -af(-bx \pm c) \pm d$	Negative coefficients, $-a$, and $-b$

Step 0 ▸ Identify the basic graph. $y = f(x)$

Step 1 ▸ Horizontal shift $y = f(x \pm c)$

Step 2 ▸ Horizontal stretch or compression $y = f(bx \pm c)$

Vertical stretch or compression $y = af(bx \pm c)$

Step 3 ▸ Reflections

Case I *none* $y = af(bx \pm c)$

Case II *about the x-axis* $y = -af(bx \pm c)$

Case III *about the y-axis* $y = af(-bx \pm c)$

First graph $y = af(bx \pm c)$, replace x by $-x$ and reflect this graph in the y-axis. Note $b(-x) = -bx$.

Case IV *about both the x-axis*
and the y-axis $y = -af(-bx \pm c)$

First graph $y = -af(bx \pm c)$, replace x by $-x$ and reflect this graph in the y-axis. Note $b(-x) = -bx$.

Step 4 ▸ Vertical shift $y = \pm af(\pm bx \pm c) \pm d$

2.9 Inverse Functions

Objectives

1 ▸ Define an inverse function.

2 ▸ Find the inverse function.

3 ▸ Use inverse functions to find the range of a function.

4 ▸ Apply inverse functions in the real world.

2.9 Inverse Functions

One-to-One Function

- If different elements in the domain of f are assigned to different values, then the function f is **one-to-one**. That is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
- If there are two distinct elements in the domain of f that are assigned the same value, then the function f is **not one-to-one**.

Let f be a one-to-one function. Then the preceding definition says that for any two numbers x_1 and x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, or equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. That is, ***f is a one-to-one function if different x-values correspond to different y-values.***

2.9 Inverse Functions

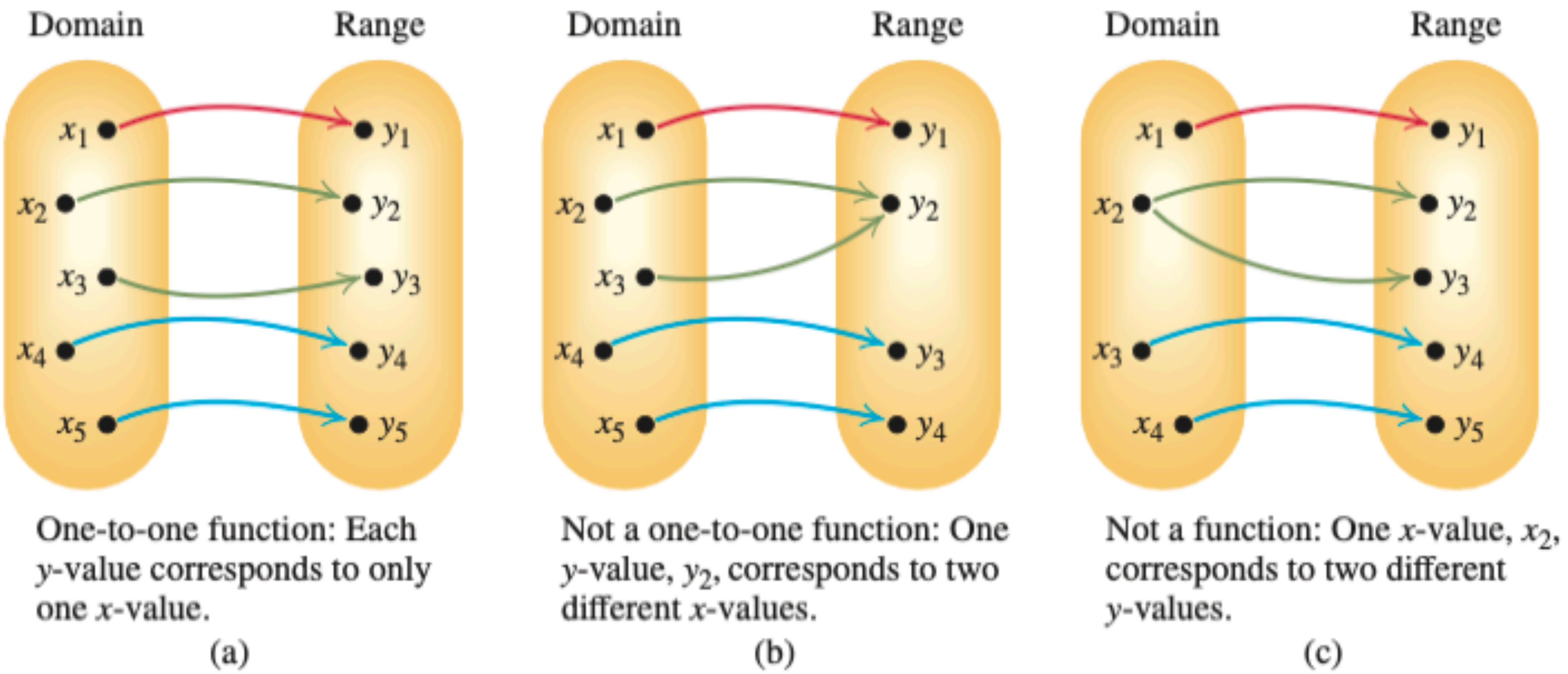


Figure 2.100 ▸ Deciding whether a diagram represents a function, a one-to-one function, or neither

2.9 Inverse Functions

Horizontal-Line Test

- If no horizontal line intersects the graph of the function more than once, then the function is **one-to-one**.
- If some horizontal line intersects the graph of the function more than once, then the function is **not one-to-one**.

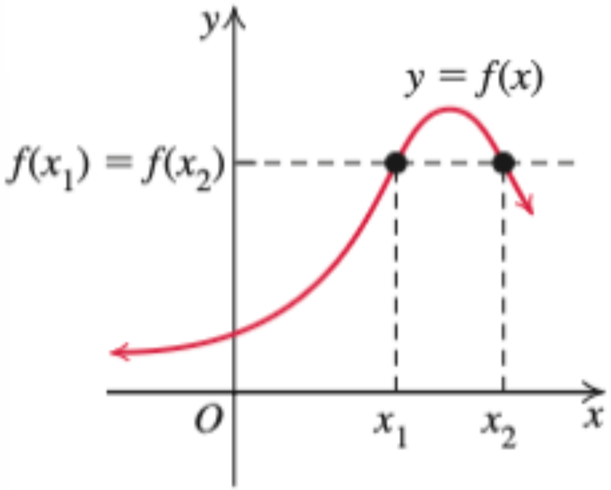


Figure 2.101 ▸ The function f is *not* one-to-one.

EXAMPLE 1 Test for the One-to-One Property of a Function

Determine which of the following functions are one-to-one.

a. $f(x) = 2x + 5$ **b.** $g(x) = x^2 - 1$ **c.** $h(x) = 2\sqrt{x}$

SIDE NOTE

The reason only a one-to-one function can have an inverse function is that if $x_1 \neq x_2$ but $f(x_1) = y$ and $f(x_2) = y$, then $f^{-1}(y)$ must be x_1 as well as x_2. This is not possible because $x_1 \neq x_2$.

EXAMPLE 1 Test for the One-to-One Property of a Function

Determine which of the following functions are one-to-one.

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Solution

Algebraic Approach Let x_1 and x_2 be two numbers in the domain of each function.

a. Suppose $f(x_1) = f(x_2)$. Then $2x_1 + 5 = 2x_2 + 5$

$$2x_1 = 2x_2 \qquad \text{Subtract 5 from each side.}$$

$$x_1 = x_2. \qquad \text{Divide both sides by 2.}$$

So $f(x_1) = f(x_2)$ implies $x_1 = x_2$; the function f is one-to-one.

b. Suppose $g(x_1) = g(x_2)$. Then $x_1^2 - 1 = x_2^2 - 1$

$$x_1^2 = x_2^2 \qquad \text{Add 1 to both sides.}$$

$$x_1 = \pm x_2. \qquad \text{Square-root property}$$

This means that for two distinct numbers, x_1 and $x_2 = -x_1$, $g(x_1) = g(x_2)$; so g is not a one-to-one function.

c. Suppose $h(x_1) = h(x_2)$. Then $2\sqrt{x_1} = 2\sqrt{x_2}$

$$\sqrt{x_1} = \sqrt{x_2} \qquad \text{Divide both sides by 2.}$$

$$x_1 = x_2. \qquad \text{Square both sides.}$$

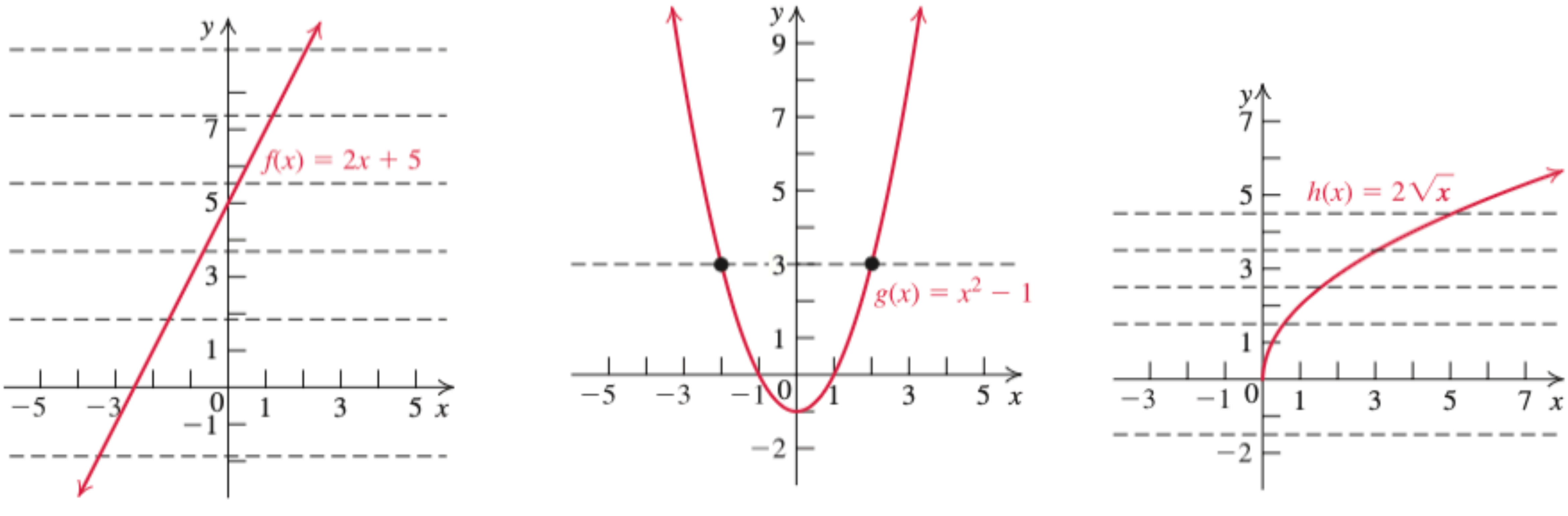
So $h(x_1) = h(x_2)$ implies $x_1 = x_2$; the function h is one-to-one.

EXAMPLE 1 **Test for the One-to-One Property of a Function**

Determine which of the following functions are one-to-one.

a. $f(x) = 2x + 5$ **b.** $g(x) = x^2 - 1$ **c.** $h(x) = 2\sqrt{x}$

Geometric Approach (Horizontal-line test) Figure 2.102 shows the graphs of functions f, g, and h. We see that no horizontal line intersects the graphs of f (Figure 2.102(a)) or h (Figure 2.102(c)) in more than one point; therefore, the functions f and h are one-to-one. The function g is not one-to-one because the horizontal line $y = 3$ (among others) in Figure 2.102(b) intersects the graph of g at more than one point.



One-to-one (a) Not one-to-one (b) One-to-one (c)

2.9 Inverse Functions

Inverse Function

Let f represent a one-to-one function. Then if y is in the range of f, there is only one value of x in the domain of f such that $f(x) = y$. We define the inverse of f, called the **inverse function of** f, denoted f^{-1}, by $f^{-1}(y) = x$ if and only if $y = f(x)$. See the input–output diagram on page 290.

From this definition we have the following:

Domain of f = Range of f^{-1} and **Range of f = Domain of f^{-1}.**

The notation $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$. The expression $\frac{1}{f(x)}$ represents the reciprocal of $f(x)$ and is sometimes written as $[f(x)]^{-1}$.

Note the input–output diagram in the margin and Figure 2.103 for $f^{-1} \circ f$.

For the graphical interpretation of the input–output diagram, see Figure 2.103.

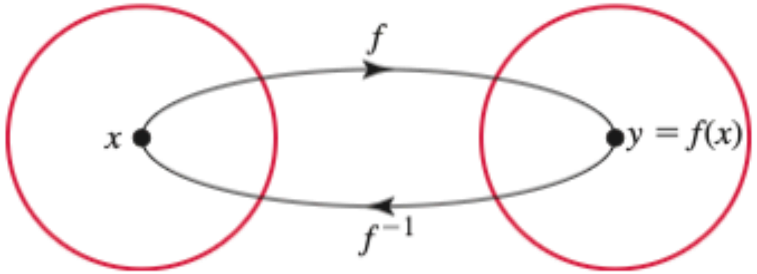


Figure 2.103 ▸ $f^{-1}(f(x)) = x$

Input–Output Diagram

$y = f(x)$ $\qquad$ $f^{-1}(y) = x$

$\underset{x}{\text{input}} \xrightarrow{f} \underset{y}{\text{output}}$ $\qquad$ $\underset{y}{\text{input}} \xrightarrow{f^{-1}} \underset{x}{\text{output}}$

$\underset{x}{\text{input}} \xrightarrow{\quad f^{-1} \circ f \quad} \underset{x}{\text{output}}$

Figure 2.103 suggests the following properties of inverse functions.

Inverse Function Property

Let f denote a one-to-one function. Then

1. $f^{-1}(f(x)) = x$ for every x in the domain of f.
2. $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.

Further, if g is any function such that, for all values of x in the domain of the inner function,

$$f(g(x)) = x \text{ and } g(f(x)) = x, \text{ then } g = f^{-1}.$$

One interpretation of the equation $(f^{-1} \circ f)(x) = x$ is that f^{-1} reverses the effect that f has on x. For example, let

2.9 Inverse Functions

One interpretation of the equation $(f^{-1} \circ f)(x) = x$ is that f^{-1} reverses the effect that f has on x. For example, let

$$f(x) = x + 2. \qquad f \text{ adds 2 to any input } x.$$

To undo what f does to x, we should subtract 2 from x. That is, the inverse of f should be

$$g(x) = x - 2. \qquad g \text{ subtracts 2 from } x.$$

Let's verify that $g(x) = x - 2$ is indeed the inverse function of x.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\
&= f(x - 2) && \text{Replace } g(x) \text{ with } x - 2. \\
&= (x - 2) + 2 && \text{Replace } x \text{ with } x - 2 \text{ in } f(x) = x + 2. \\
&= x && \text{Simplify.}
\end{aligned}$$

We leave it for you to check that $(g \circ f)(x) = x$.

2.9 Inverse Functions

EXAMPLE 2 **Relating the Values of a Function and Its Inverse**

Assume that f is a one-to-one function.

a. If $f(3) = 5$, find $f^{-1}(5)$.

b. If $f^{-1}(-1) = 7$, find $f(7)$.

2.9 Inverse Functions

EXAMPLE 2 Relating the Values of a Function and Its Inverse

Assume that f is a one-to-one function.

a. If $f(3) = 5$, find $f^{-1}(5)$. **b.** If $f^{-1}(-1) = 7$, find $f(7)$.

Solution

By definition, $f^{-1}(y) = x$ if and only if $y = f(x)$.

a. Let $x = 3$ and $y = 5$. Now reading the definition from right to left, $5 = f(3)$ if and only if $f^{-1}(5) = 3$. So $f^{-1}(5) = 3$.

b. Let $y = -1$ and $x = 7$. Now $f^{-1}(-1) = 7$ if and only if $f(7) = -1$. So $f(7) = -1$.

2.9 Inverse Functions

EXAMPLE 3 Verifying Inverse Functions

Verify that the following pairs of functions are inverses of each other:

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = \frac{x - 3}{2}.$$

2.9 Inverse Functions

EXAMPLE 3 Verifying Inverse Functions

Verify that the following pairs of functions are inverses of each other:

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = \frac{x - 3}{2}.$$

Solution

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\
&= f\left(\frac{x-3}{2}\right) && \text{Replace } g(x) \text{ with } \frac{x-3}{2}. \\
&= 2\left(\frac{x-3}{2}\right) + 3 && \text{Replace } x \text{ with } \frac{x-3}{2} \text{ in } f(x) = 2x + 3. \\
&= x && \text{Simplify.}
\end{aligned}$$

2.9 Inverse Functions

Thus, $f(g(x)) = x.$

$$
\begin{aligned}
g(x) &= \frac{x-3}{2} && \text{Given function } g \\
(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \text{ of } f \\
&= g(2x+3) && \text{Replace } f(x) \text{ with } 2x+3. \\
&= \frac{(2x+3)-3}{2} && \text{Replace } x \text{ with } 2x+3 \text{ in } g(x) = \frac{x-3}{2}. \\
&= x && \text{Simplify.}
\end{aligned}
$$

Thus, $g(g(x)) = x$. Because $f(g(x)) = g(f(x)) = x$, f and g are inverses of each other.

Practice Problem 3. Verify that $f(x) = 3x - 1$ and $g(x) = \frac{x+1}{3}$ are inverses of each other.

2.9 Inverse Functions

In Example 3, notice how the functions f and g reverse (undo) the effect of each other. The function f takes an input x, ***multiplies*** it by 2, and ***adds*** 3; g reverses (or undoes) this effect by ***subtracting*** 3 and ***dividing*** by 2. This process is illustrated in Figure 2.104. Notice that g reverses the operations performed by f *and* the order in which they are done.

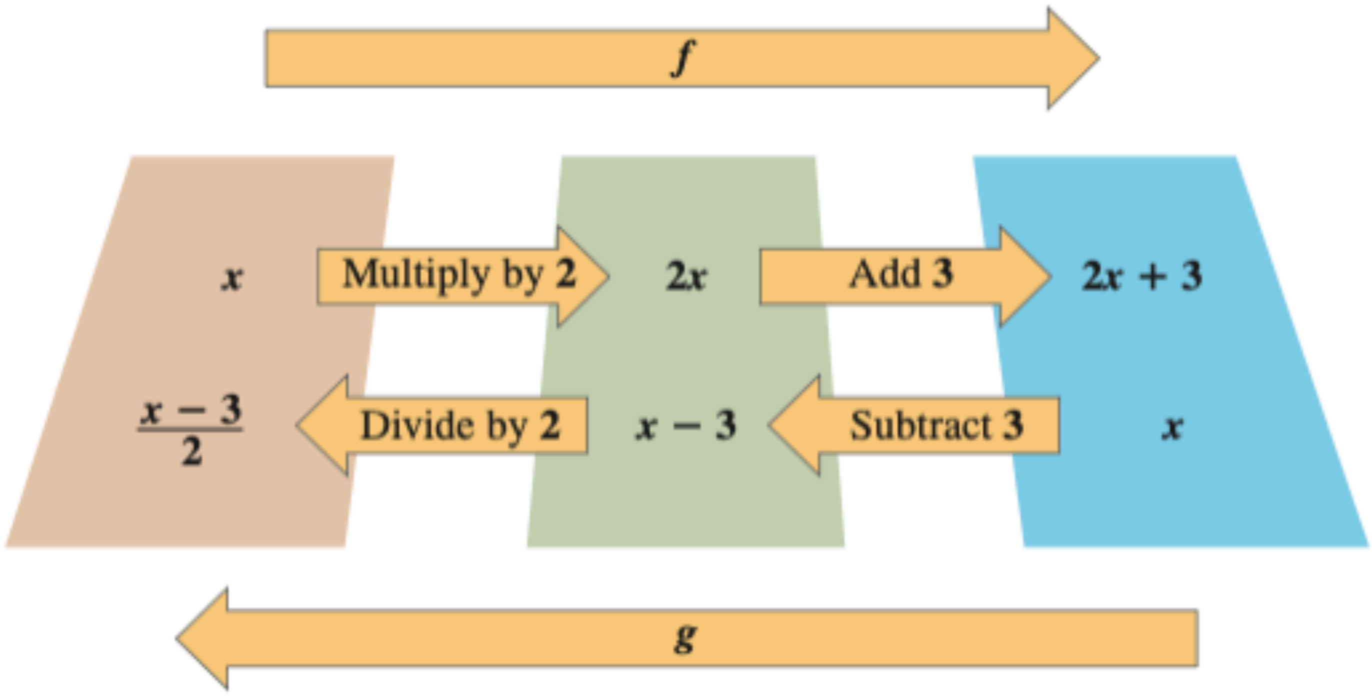


Figure 2.104 ▸ Function g undoes f.

2.9 Inverse Functions

SIDE NOTE

Interchanging x and y in Step 2 is done so that we can write f^{-1} in the usual format, with x as the independent variable and y as the dependent variable.

PROCEDURE IN ACTION

EXAMPLE 5 Finding an Equation for f^{-1}

Objective	Example
Find the inverse of a one-to-one function f.	*Find the inverse of* $f(x) = 3x - 4$.
Step 1 ▸ Replace $f(x)$ with y in the equation defining $f(x)$.	**1.** $y = 3x - 4$ Replace $f(x)$ with y.
Step 2 ▸ Interchange x and y.	**2.** $x = 3y - 4$ Interchange x and y.
Step 3 ▸ Solve the equation in Step 2 for y.	**3.** $x + 4 = 3y$ Add 4 to both sides. $\dfrac{x+4}{3} = y$ Divide both sides by 3.
Step 4 ▸ Write $f^{-1}(x)$ for y.	**4.** $f^{-1}(x) = \dfrac{x+4}{3}$ We usually end with $f^{-1}(x)$ on the left.

Practice Problem 5. Find the inverse of $f(x) = -2x + 3$. ■

2.9 Inverse Functions

EXAMPLE 6 Finding the Inverse Function

Find the inverse of the one-to-one function

$$f(x) = \frac{x+1}{x-2}, \quad x \neq 2.$$

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Find the inverse of the one-to-one function

$$f(x) = \frac{x+1}{x-2}, \quad x \neq 2.$$

Solution

Step 1 ▸ $y = \dfrac{x+1}{x-2}$ Replace $f(x)$ with y.

Step 2 ▸ $x = \dfrac{y+1}{y-2}$ Interchange x and y.

Step 3 ▸ Solve $x = \dfrac{y+1}{y-2}$ for y. This is the most challenging step.

$$
\begin{aligned}
x(y-2) &= y+1 && \text{Multiply both sides by } y-2.\\
xy - 2x &= y+1 && \text{Distributive property}\\
xy - 2x + 2x - y &= y + 1 + 2x - y && \text{Add } 2x - y \text{ to both sides.}\\
xy - y &= 2x + 1 && \text{Simplify.}\\
y(x-1) &= 2x+1 && \text{Factor out } y.\\
y &= \frac{2x+1}{x-1} && \text{Divide both sides by } x-1, \text{ assuming that } x \neq 1.
\end{aligned}
$$

Step 4 ▸ $f^{-1}(x) = \dfrac{2x+1}{x-1}, \; x \neq 1$ Replace y with $f^{-1}(x)$.

To see if our calculations are accurate, we compute $f(f^{-1}(x))$ and $f^{-1}(f(x))$.

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x+1}{x-2}\right) = \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \frac{2x+2+x-2}{x+1-x+2} = \frac{3x}{3} = x$$

Multiply numerator and denominator by $x - 2$ and simplify.

You should also check that $f(f^{-1}(x)) = x$.

2.9 Inverse Functions

EXAMPLE 8 Finding an Inverse Function

Find the inverse of $G(x) = x^2 - 1$, $x \geq 0$.

2.9 Inverse Functions

EXAMPLE 8 **Finding an Inverse Function**

Find the inverse of $G(x) = x^2 - 1,\ x \geq 0$.

Solution

Step 1 ▸ $y = x^2 - 1,\ x \geq 0$ — Replace $G(x)$ with y.

Step 2 ▸ $x = y^2 - 1,\ y \geq 0$ — Interchange x and y.

Step 3 ▸ $x + 1 = y^2,\ y \geq 0$ — Add 1 to both sides.

$y = \sqrt{x+1}$, because $y \geq 0$ — Solve for y.

Step 4 ▸ $G^{-1}(x) = \sqrt{x+1}$ — Replace y with $G^{-1}(x)$.

The graphs of G and G^{-1} are shown in Figure 2.110.

2.9 Inverse Functions

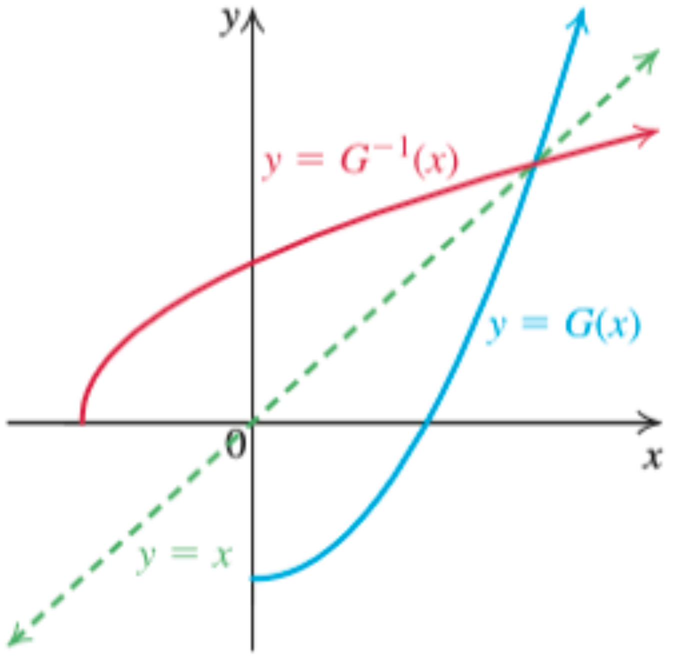

Figure 2.110 ▸ Graphs of G and G^{-1}

EXAMPLE 8 Finding an Inverse Function

Find the inverse of $G(x) = x^2 - 1,\ x \geq 0$.

Solution

Step 1 ▸ $y = x^2 - 1,\ x \geq 0$ — Replace $G(x)$ with y.

Step 2 ▸ $x = y^2 - 1,\ y \geq 0$ — Interchange x and y.

Step 3 ▸ $x + 1 = y^2,\ y \geq 0$ — Add 1 to both sides.

$y = \sqrt{x + 1}$, because $y \geq 0$ — Solve for y.

Step 4 ▸ $G^{-1}(x) = \sqrt{x + 1}$ — Replace y with $G^{-1}(x)$.

The graphs of G and G^{-1} are shown in Figure 2.110.

2.9 Inverse Functions

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

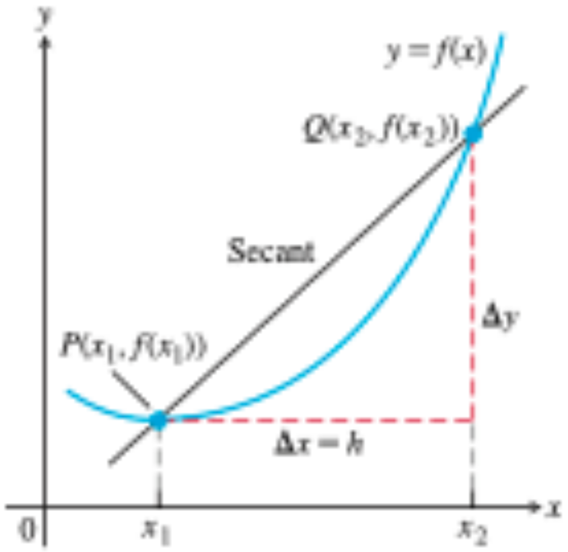

FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant line PQ. As the point Q approaches the point P along the curve, the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

1.1 sectio

PROCEDURE IN ACTION

EXAMPLE 3 **The Substitution Method**

Objective	**Example**
Reduce the solution of the system to the solution of one equation in one variable by substitution.	*Solve the system:* $\begin{cases} 2x - 5y = 3 & (1) \\ y - 2x = 9 & (2). \end{cases}$
Step 1 ▸ Solve for one variable. Choose one of the equations and express one of its variables in terms of the other variable.	**1.** We choose equation (2) and express y in terms of x. $y = 2x + 9 \quad (3)$ — Solve equation (2) for y.
Step 2 ▸ Substitute. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.	**2.** $2x - 5y = 3$ — Equation (1) $2x - 5(2x + 9) = 3$ — Substitute $2x + 9$ for y. $2x - 10x - 45 = 3$ — Distributive property
Step 3 ▸ Solve the equation obtained in Step 2.	**3.** $-8x - 45 = 3$ — Combine like terms. $-8x = 3 + 45$ — Add 45 to both sides. $x = -6$ — Solve for x.
Step 4 ▸ Back-substitute. Substitute the value(s) you found in Step 3 back into the expression you found in Step 1. The result is the solution(s).	**4.** $y = 2x + 9$ — Equation (3) from Step 1 $y = 2(-6) + 9 = -3$ — Substitute $x = -6$. Because $x = -6$ and $y = -3$, the solution set is $\{(-6, -3)\}$.
Step 5 ▸ Check. Check your answer(s) in the original equations.	**5.** Check: $x = -6$ and $y = -3$. Equation (1): $2(-6) - 5(-3) \stackrel{?}{=} 3$; $-12 + 15 = 3$ ✓ Equation (2): $-3 - 2(-6) \stackrel{?}{=} 9$; $-3 + 12 = 9$ ✓

Practice Problem 3. Solve: $\begin{cases} x - y = 5 & (1) \\ 2x + y = 7 & (2). \end{cases}$

1.1 section review.

104. **Investment.** Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

1.1 section.

104. Investment. Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1) + y (investment 2) = ?

1.1 section.

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- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.

8.1 section.

104. Investment. Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.
- .08x + .105y = \$2550.
- Then: 8x + 10.5y = \$255000.

8.1 section.

104. Investment. Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.
- .08x + .105y = \$2550.
- Then: 8x + 10.5y = \$255000.
- Substitute: 8(\$30000 – y) + 10.5y = \$255000.
- \$240000 – 8y + 10.5y = \$255000.
- 2.5y = \$15000.
- y = \$6000.
- Substitute into first, total equation:
- x + \$6000 = \$30000.
- x = \$24000.

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$

$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

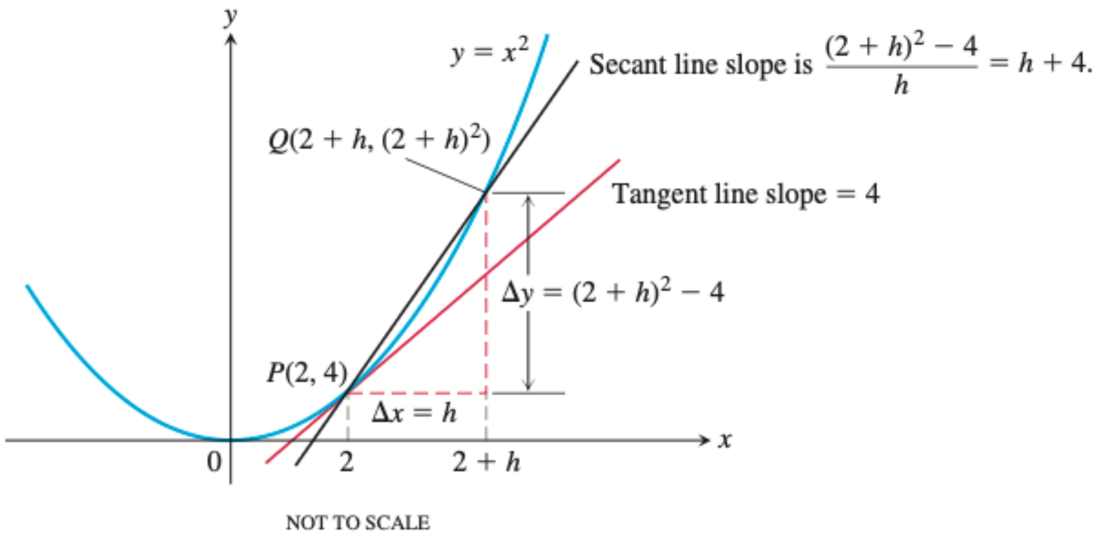

FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

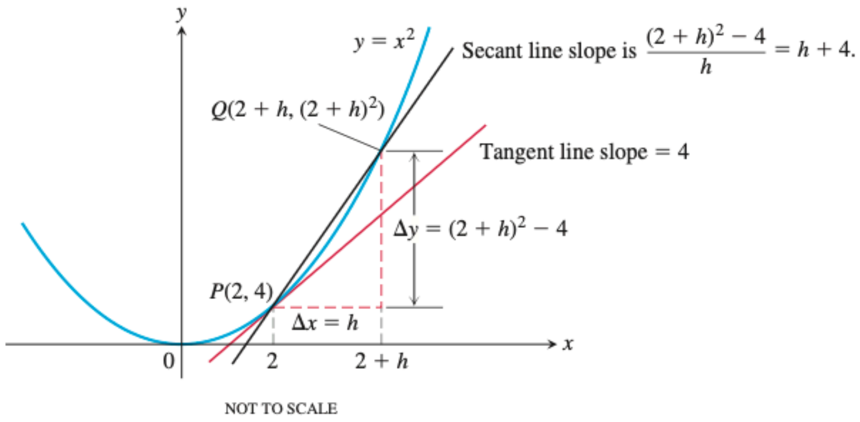

FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$
$$y = 4x - 4.$$

Slope of a Curve at a Point

In Exercises 7–18, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

7. $y = x^2 - 5, \quad P(2, -1)$

8. $y = 7 - x^2, \quad P(2, 3)$

9. $y = x^2 - 2x - 3, \quad P(2, -3)$

10. $y = x^2 - 4x, \quad P(1, -3)$

11. $y = x^3, \quad P(2, 8)$

12. $y = 2 - x^3, \quad P(1, 1)$

13. $y = x^3 - 12x, \quad P(1, -11)$

14. $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

15. $y = \dfrac{1}{x}, \quad P(-2, -1/2)$

An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function f at an interior point of the domain of f. Suppose that $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L." In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c. The value of the function at c itself is not considered.

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

THEOREM 1—Limit Laws

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

EXAMPLE 5 Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\displaystyle\lim_{x\to c}(x^3 + 4x^2 - 3)$

(b) $\displaystyle\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\displaystyle\lim_{x\to -2}\sqrt{4x^2 - 3}$

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Solution

(a)

$$\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 \qquad \text{Sum and Difference Rules}$$

$$= c^3 + 4c^2 - 3 \qquad \text{Power and Multiple Rules}$$

(b)

$$\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)} \qquad \text{Quotient Rule}$$

$$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5} \qquad \text{Sum and Difference Rules}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \qquad \text{Power or Product Rule}$$

(c)

$$\lim_{x\to -2}\sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2}(4x^2 - 3)} \qquad \text{Root Rule with } n = 2$$

$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} \qquad \text{Difference Rule}$$

$$= \sqrt{4(-2)^2 - 3} \qquad \text{Product and Multiple Rules and limit of a constant function}$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

■

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to-3}(x^2 - 13)$

12. $\lim_{x\to2}(-x^2 + 5x - 2)$

13. $\lim_{t\to6} 8(t - 5)(t - 7)$

14. $\lim_{x\to-2}(x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to2}\dfrac{2x + 5}{11 - x^3}$

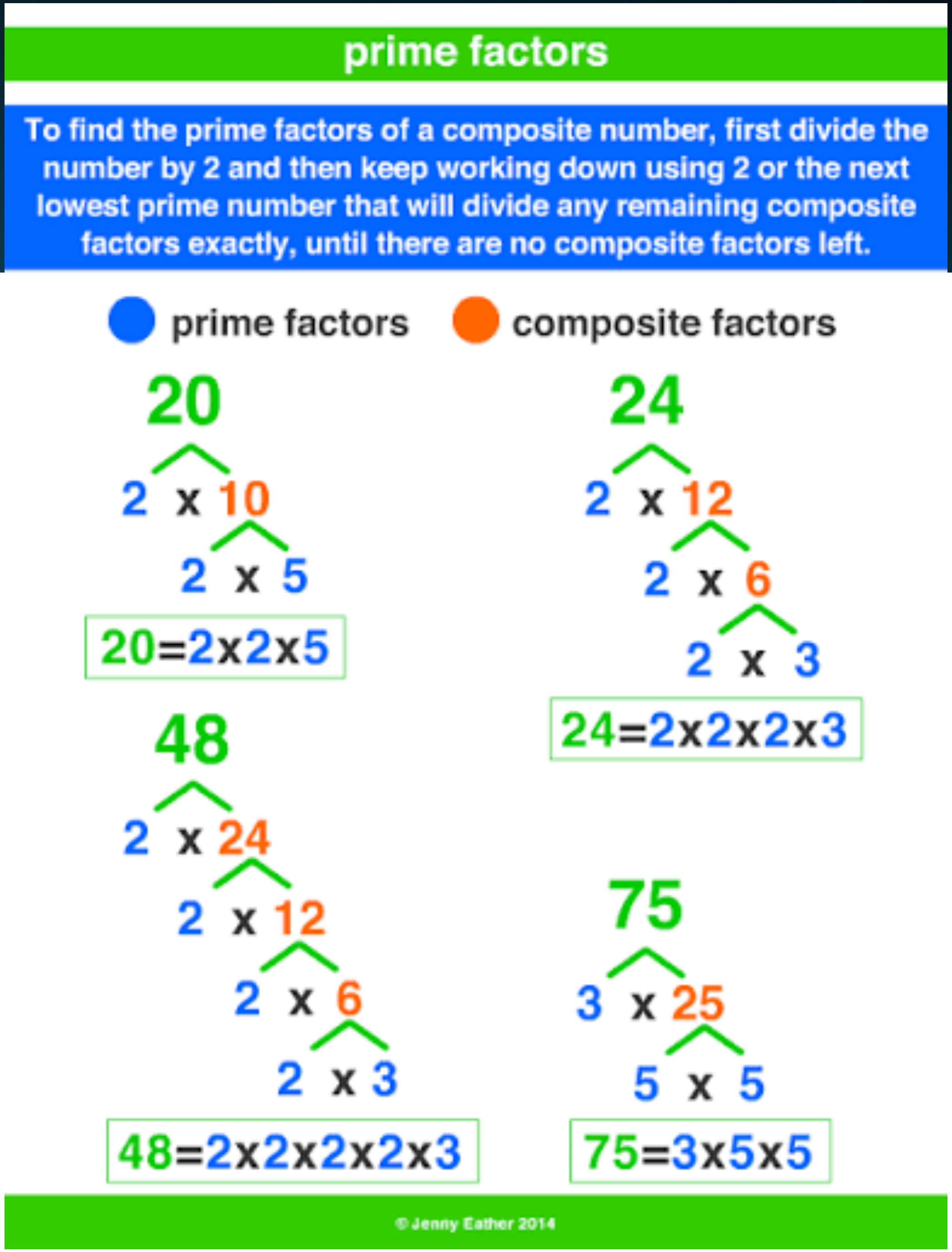
16. $\lim_{s\to2/3}(8 - 3s)(2s - 1)$

17. $\lim_{x\to-1/2} 4x(3x + 4)^2$

18. $\lim_{y\to2}\dfrac{y + 2}{y^2 + 5y + 6}$

19. $\lim_{y\to-3}(5 - y)^{4/3}$

20. $\lim_{z\to4}\sqrt{z^2 - 10}$


prime factors
To find the prime factors of a composite number, first divide the number by 2 and then keep working down using 2 or the next lowest prime number that will divide any remaining composite factors exactly, until there are no composite factors left.
prime factors
composite factors
20
2 x 10
2 x 5
20=2x2x5
24
2 x 12
2 x 6
2 x 3
24=2x2x2x3
48
2 x 24
2 x 12
2 x 6
2 x 3
48=2x2x2x2x3
75
3 x 25
5 x 5
75=3x5x5
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P6 section review.

Square Root Property

For $a \geq 0$, if $x^2 = a$ then $x = \pm\sqrt{a}$.

The Exponent $\frac{1}{n}$

For any real number a and any integer $n > 1$,

$$a^{1/n} = \sqrt[n]{a}.$$

When n is even and $a < 0$, $\sqrt[n]{a}$ and $a^{1/n}$ are not real numbers.
Notice that the denominator n of the rational exponent is the index of the radical.

P6 section review.

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Exponents and Radicals

$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn} \qquad (ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad a^{-n} = \frac{1}{a^n}$$

If n is even, $\sqrt[n]{a^n} = |a|$.

If n is odd, $\sqrt[n]{a^n} = a$.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \quad a, b \geq 0$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$$

P6 section review.

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$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$

WARNING

In general:

$\sqrt[n]{a^n + b^n} \neq a + b$

$\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

$\sqrt[n]{a} + \sqrt[n]{b}$ cannot be added if $a \neq b$.

$\sqrt[n]{a} + \sqrt[n]{a} = 2\sqrt[n]{a}$

$\sqrt[n]{a} + \sqrt[m]{a}$ cannot be added if $n \neq m$.

P6 section review.

The Exponent $\frac{1}{n}$ and Radicals

If a is any real number and n is any integer greater than 1, then

	***n* even**	***n* odd**
If $a > 0$,	$\sqrt[n]{a} = a^{1/n}$ is positive.	$\sqrt[n]{a} = a^{1/n}$ is positive.
If $a < 0$,	$\sqrt[n]{a} = a^{1/n}$ is not a real number.	$\sqrt[n]{a} = a^{1/n}$ is negative.
If $a = 0$,	$\sqrt[n]{0} = 0^{1/n} = 0$.	$\sqrt[n]{0} = 0^{1/n} = 0$.

Rational Exponents

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m},$$

provided that m and n are integers with no common factors, $n > 1$, and $\sqrt[n]{a}$ is a real number.

P6 section review.

Properties of Rational Exponents

If r and s are rational numbers and a and b are real numbers, then

$$a^r \cdot a^s = a^{r+s} \qquad (a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r$$

$$\frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad a^{-r} = \frac{1}{a^r} \qquad \left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^r$$

provided that all of the expressions used are defined.

P6 section review.

P6 section review.

Conjugates

Pairs of expressions of the form $a\sqrt{x} + b\sqrt{y}$ and $a\sqrt{x} - b\sqrt{y}$ are called conjugates. We form the product of these two conjugates.

$$\begin{aligned}(a\sqrt{x} + b\sqrt{y})(a\sqrt{x} - b\sqrt{y}) &= (a\sqrt{x})^2 - (b\sqrt{y})^2 && \text{Difference of two squares}\\ &= a^2x - b^2y && (ac)^2 = a^2c^2;\ (\sqrt{z})^2 = z\end{aligned}$$

We see that the product contains no radicals. So the product of conjugates can be used to rationalize the denominator or the numerator.

P6 section review.

P6 sec

EXAMPLE 6 Adding and Subtracting Radical Expressions

Simplify: **a.** $\sqrt{45} + 7\sqrt{20}$. **b.** $5\sqrt[3]{80x} - 3\sqrt[3]{270x}$.

Solution

a. We find perfect-square factors for both 45 and 20.

$$\begin{aligned}
\sqrt{45} + 7\sqrt{20} &= \sqrt{9 \cdot 5} + 7\sqrt{4 \cdot 5} && 45 = 9 \cdot 5;\ 20 = 4 \cdot 5 \\
&= \sqrt{9}\sqrt{5} + 7\sqrt{4}\sqrt{5} && \text{Product rule} \\
&= 3\sqrt{5} + 7 \cdot 2\sqrt{5} && \sqrt{9} = 3;\ \sqrt{4} = 2 \\
&= 3\sqrt{5} + 14\sqrt{5} && \text{Like radicals} \\
&= (3 + 14)\sqrt{5} && \text{Distributive property} \\
&= 17\sqrt{5}
\end{aligned}$$

b. This time we find perfect-cube factors for both 80 and 270.

$$\begin{aligned}
5\sqrt[3]{80x} - 3\sqrt[3]{270x} &= 5\sqrt[3]{8 \cdot 10x} - 3\sqrt[3]{27 \cdot 10x} && 80 = 8 \cdot 10;\ 270 = 27 \cdot 10 \\
&= 5\sqrt[3]{8} \cdot \sqrt[3]{10x} - 3\sqrt[3]{27} \cdot \sqrt[3]{10x} && \text{Product rule} \\
&= 5 \cdot 2 \cdot \sqrt[3]{10x} - 3 \cdot 3\sqrt[3]{10x} && \sqrt[3]{8} = 2;\ \sqrt[3]{27} = 3 \\
&= 10\sqrt[3]{10x} - 9\sqrt[3]{10x} && \text{Like radicals} \\
&= (10 - 9)\sqrt[3]{10x} && \text{Distributive property} \\
&= \sqrt[3]{10x}
\end{aligned}$$

Practice Problem 6. Simplify.

a. $3\sqrt{12} + 7\sqrt{3}$ **b.** $2\sqrt[3]{135x} - 3\sqrt[3]{40x}$

Motivation.

Introduce ourselves.

Motivation.

What is mathematics?

Why should we study further, and gain proficiency, in mathematics?

Motivation.

What is mathematics?

What is math?

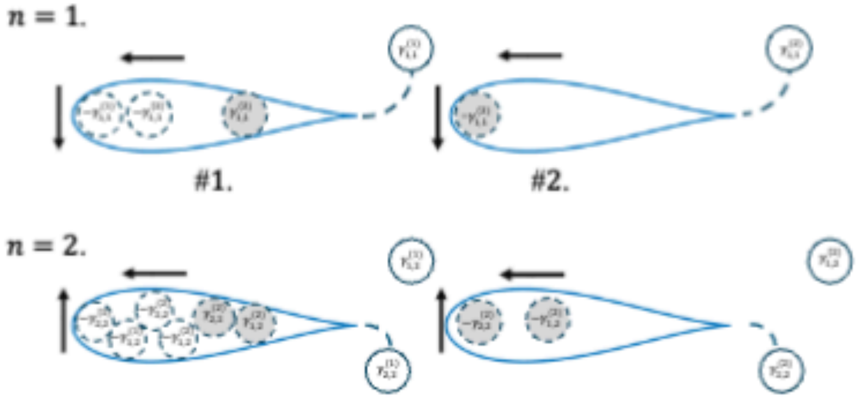


Figure 2.3 Schematic of the flow-induced model configuration for $M = 2$ wings and their shed vortices in two time steps ($n = 1$ and $n = 2$). The velocity potential around wing 1 is considered, so the second wing is only represented by its bound vorticity (gray circles). The dotted circles represent image vortices. The dashed curves connect the trailing edges of each wing to the vortex most recently shed. The bound vorticity before the vortex is shed is $\Gamma_{1,1}^{(1)} = 0$, $\Gamma_{1,1}^{(2)} = 0$.

step is

$$
\begin{aligned}
w_n^{(j)}(\zeta) = {} & \overline{W}_n^{(j)}(J(\zeta)-\zeta) - \frac{r^2 W_n^{(j)}}{\zeta} - i\frac{1}{2\pi}\sum_{l=1}^{M}\sum_{k=1}^{n}\gamma_{k,n}^{(l)}\log\left(\frac{-r}{\zeta_{k,n}^{(j,l)}}\frac{\zeta-\zeta_{k,n}^{(j,l)}}{\zeta-r^2/\overline{\zeta}_{k,n}^{(j,l)}}\right) \\
& - i\frac{1}{2\pi}\sum_{l\neq j}^{M}\left(\Gamma_n^{(l)}-\gamma_n^{(l)}\right)\log\left(\frac{-r}{\chi_n^{(j,l)}}\frac{\zeta-\chi_n^{(j,l)}}{\zeta-r^2/\overline{\chi}_n^{(j,l)}}\right) \\
& - i\frac{1}{2\pi}\left[\sum_{l=1}^{M}\left(\Gamma_n^{(l)}+\sum_{k=1}^{n-1}\gamma_{k,n}^{(l)}\right)\right]\log\left(\frac{\zeta}{r}\right), \\
\zeta_{k,n}^{(j,l)} = {} & J^{-1}\left(z_{k,n}^{(l)}-Z_n^{(j)}\right), \quad \chi_n^{(j,l)} = J^{-1}\left(Z_n^{(l)}-Z_n^{(j)}\right). \qquad (2.23)
\end{aligned}
$$

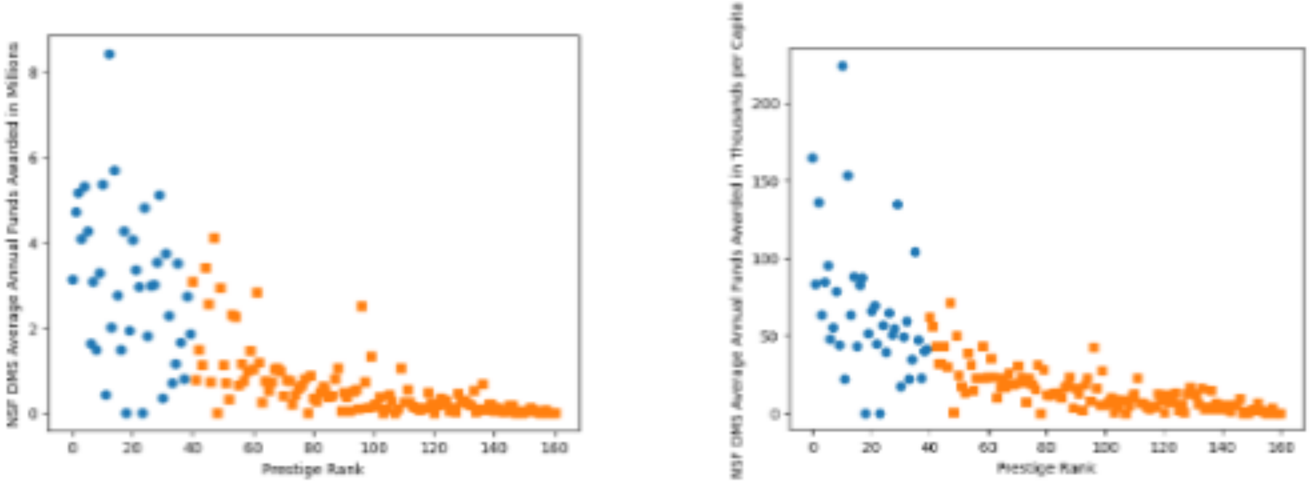
That is, $\zeta_{k,n}^{(j,l)}$ is the position of vortex $z_{k,n}^{(l)}$ at time n in the ζ-plane, which was shed by wing l at time step k, using coordinates where wing j is at the center. Similarly, $\chi_n^{(j,l)}$ is the position of wing l at time n, again using coordinates where wing j is at the

Motivation.

What is mathematics?

Quantifying & Documenting Inequalities in Math 11

12 Buckmire *et al.*



(a) Total funding

(b) Funding per faculty

Fig. 4: Average annual grant funding for Mathematics departments by the prestige rank of the department, displayed by total funding to the department (a), and on a per capita basis accounting for the varying number of faculty in each department (b).

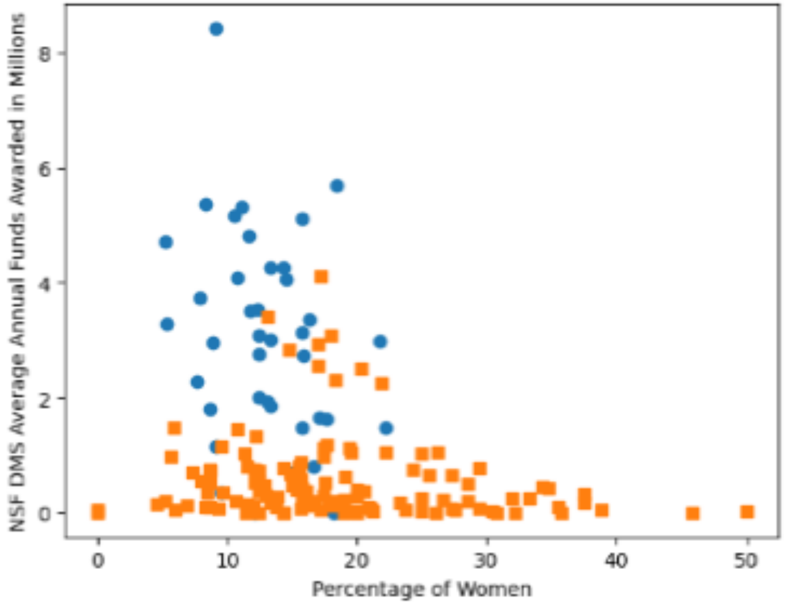


Fig. 5: Average annual grant funding for Mathematics departments by the percentage of women faculty in the department. The upper quartile (by prestige) departments are represented by blue circles, while the remainder are orange squares.

- Questions?

  Study hard, best of luck!