# ***Engineering Academy***

Dr. José L. Pabón, PhD

*Class will start soon.*

We will pass this course with a great grade & meet our academic and professional goals!

# Current students

If you see this slide, email the professor with the answer of the square root to 144, plus making a reference to this slide, for bonus points.

- Attendance is required.

  Any absence justifications must be addressed with the office of the appropriate staff .

- Reminder!

We will be courteous, civil to each other.
NO SUCH THING AS AN OBVIOUS QUESTION
ask ask ask any doubt to clear up.

- Reminder

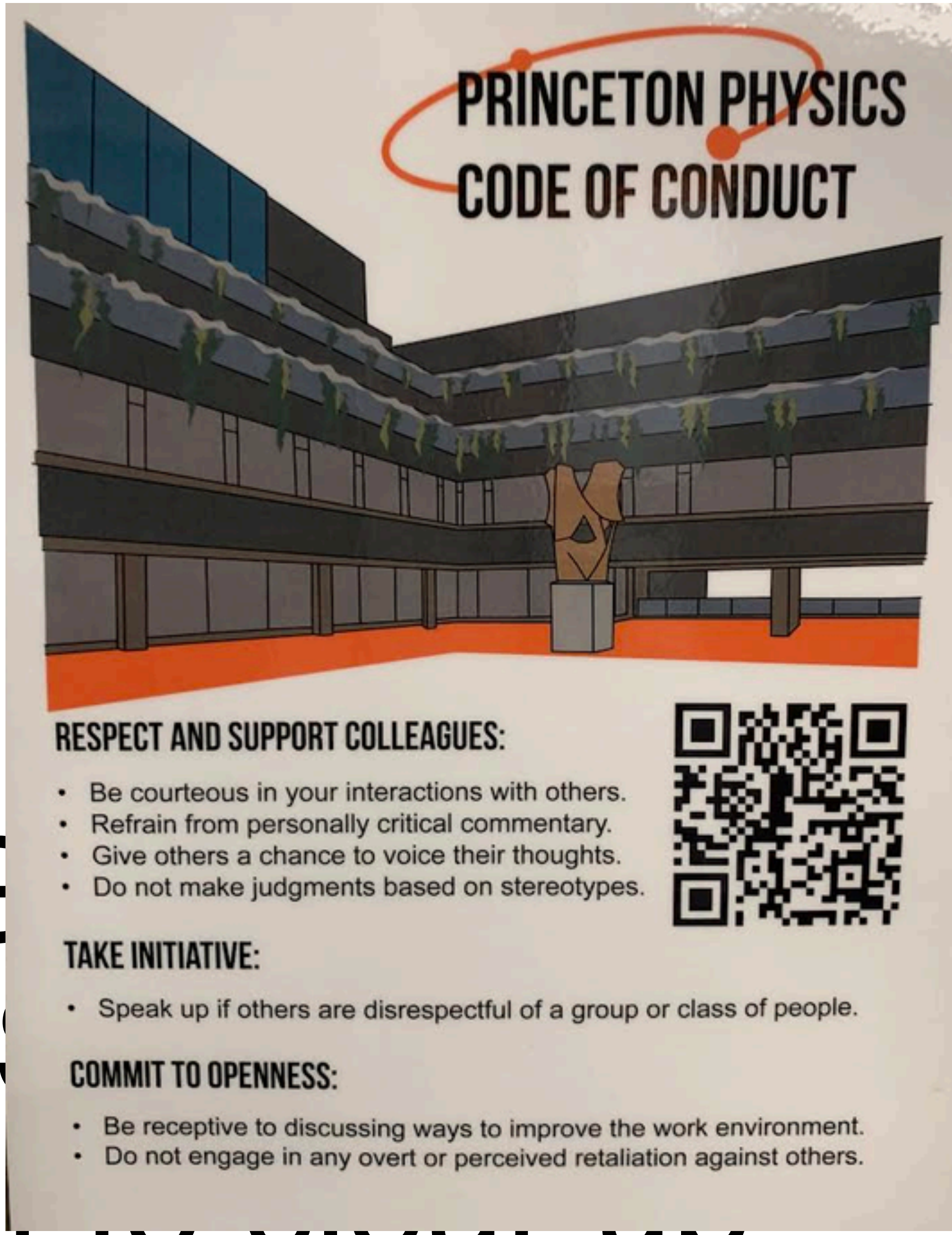
We will be courteous,

other.

NO SUCH THING

OBVIOUS QUES

ask ask ask any doubt to clear up.



# Integrity – Personal, Academic, Professional.

# P6 section review.

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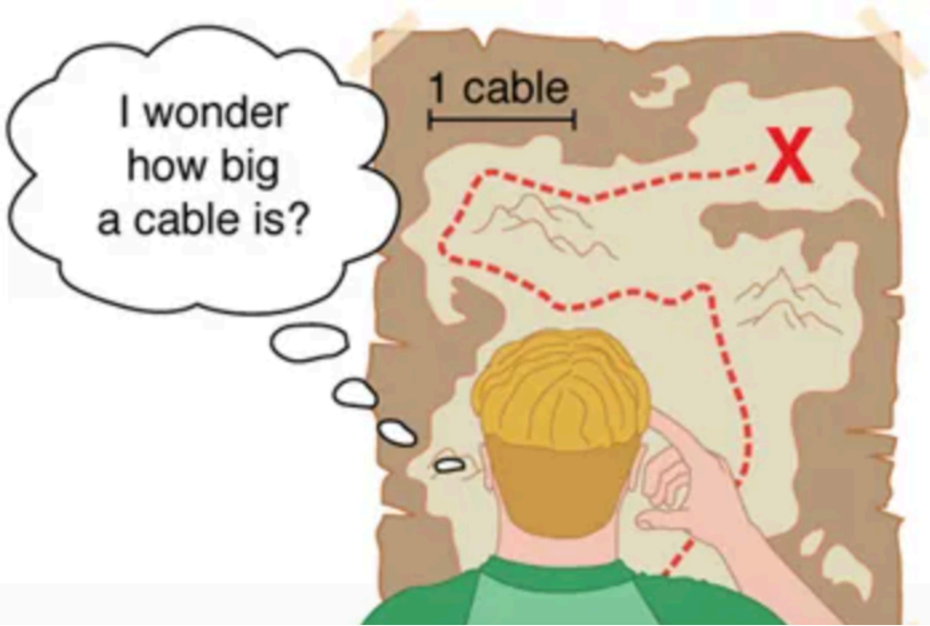
We define a **physical quantity** either by *specifying how it is measured* or by *stating how it is calculated* from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define *average speed* by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See Figure 1.16.)



There are two major systems of units used in the world: **SI units** (also known as the metric system) and **English units** (also known as the customary or imperial system). **English units** were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym “SI” is derived from the French *Système International*.

# Metric vs. Imperial

## The world's two major measurement systems

### Weight and Volume

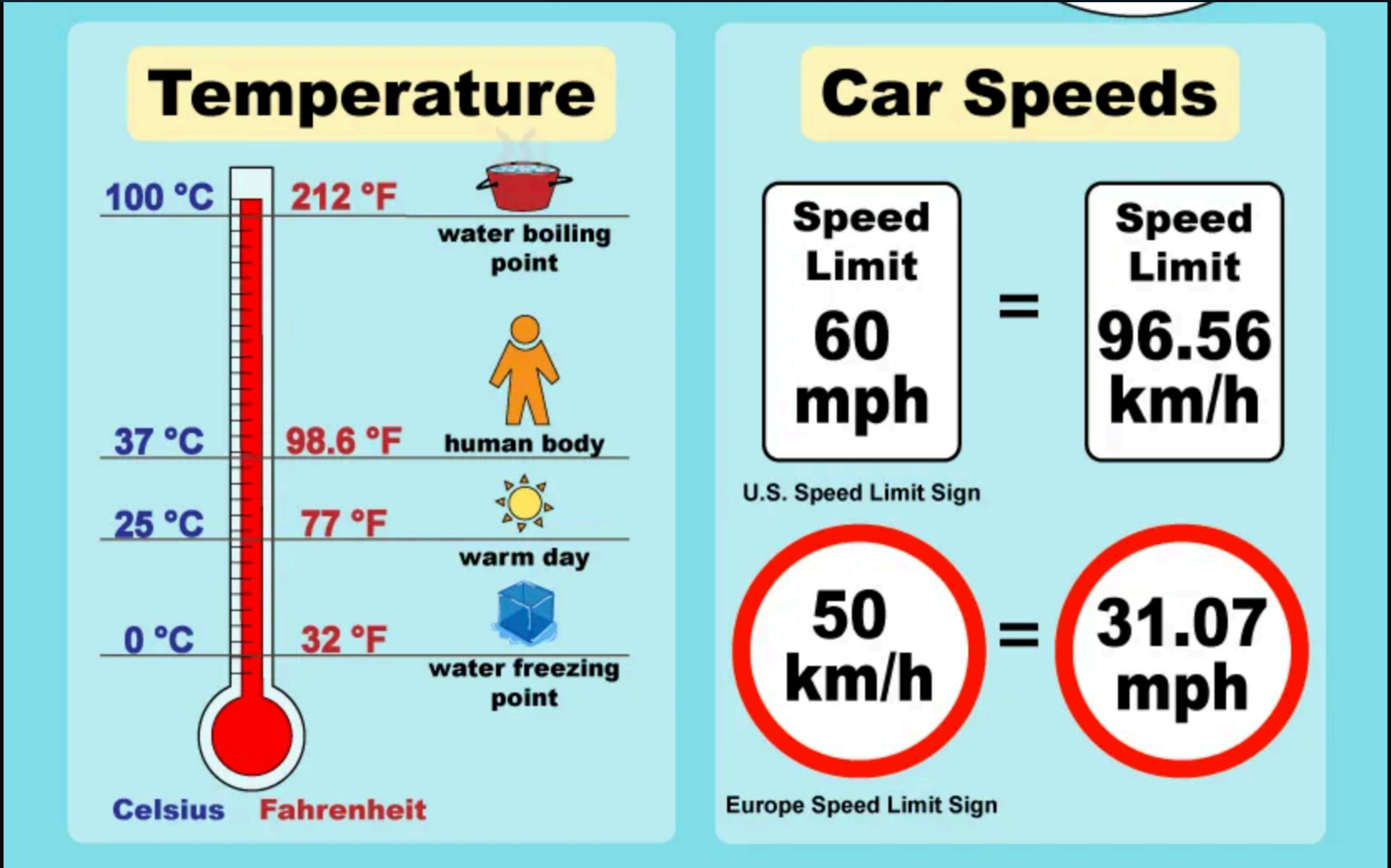
1/4 pound
113.4 grams

1/2 gallon
1.89 liters

250 grams
0.55 pounds

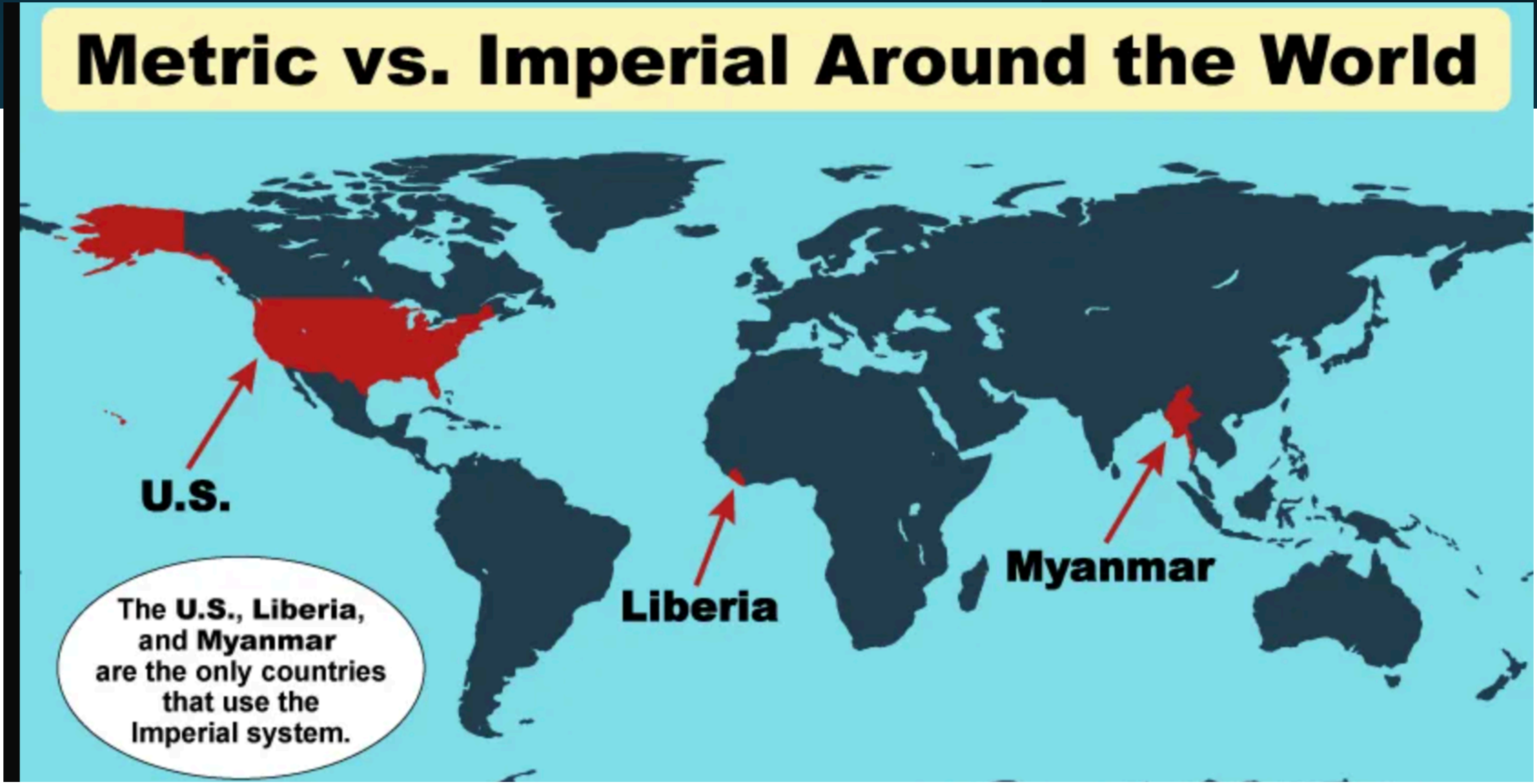
2 liters
0.53 gallons

The U.K. also uses imperial sizes for milk, including 1, 2, and 6 pints!



Temperature
100 °C
212 °F
water boiling point
37 °C
98.6 °F
human body
25 °C
77 °F
warm day
0 °C
32 °F
water freezing point
Celsius
Fahrenheit
Car Speeds
Speed Limit 60 mph
=
Speed Limit 96.56 km/h
U.S. Speed Limit Sign
50 km/h
=
31.07 mph
Europe Speed Limit Sign


Length and Height
1 cm = .39 in.
100 cm = 1 m = 39.4 in.
1 in. = 2.54 cm
12 in. = 1 ft = 30.48 cm
5 ft
152.4 cm
200 cm
6.56 ft
Standard Running Track
400 m ≈ .25 mile
4 laps = 1600 m ≈ 1 mile
Running tracks are almost universally measured in meters, but the U.S. colloquially treats 1600m as a mile.


Metric vs. Imperial Around the World
U.S.
Liberia
Myanmar
The U.S., Liberia, and Myanmar are the only countries that use the Imperial system.

| Prefix | Symbol | Value | Example (some are approximate) | | | |
|---|---|---|---|---|---|---|
| exa | E | $10^{18}$ | exameter | Em | $10^{18}$ m | distance light travels in a century |
| peta | P | $10^{15}$ | petasecond | Ps | $10^{15}$ s | 30 million years |
| tera | T | $10^{12}$ | terawatt | TW | $10^{12}$ W | powerful laser output |
| giga | G | $10^{9}$ | gigahertz | GHz | $10^{9}$ Hz | a microwave frequency |
| mega | M | $10^{6}$ | megacurie | MCi | $10^{6}$ Ci | high radioactivity |

| | |
|---|---|
| Length | $1 \text{ inch (in.)} = 2.54 \text{ cm (exactly)}$ |
| | $1 \text{ foot (ft)} = 0.3048 \text{ m}$ |
| | $1 \text{ mile (mi)} = 1.609 \text{ km}$ |
| Force | $1 \text{ pound (lb)} = 4.448 \text{ N}$ |
| Energy | $1 \text{ British thermal unit (Btu)} = 1.055 \times 10^3 \text{ J}$ |
| Power | $1 \text{ horsepower (hp)} = 746 \text{ W}$ |
| Pressure | $1 \text{ lb/in}^2 = 6.895 \times 10^3 \text{ Pa}$ |

# Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, $A$, is often denoted as $\delta A$ ("delta $A$"), so the measurement result would be recorded as $A \pm \delta A$. In our paper example, the length of the paper could be expressed as $11\ \text{in.} \pm 0.2$.

## Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement $A$ is expressed with uncertainty, $\delta A$, the **percent uncertainty** (%unc) is defined to be

$$\%\ \text{unc} = \frac{\delta A}{A} \times 100\%. \tag{1.7}$$

A grocery store sells $5\text{-lb}$ bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

Week 1 weight: $4.8\ \text{lb}$

Week 2 weight: $5.3\ \text{lb}$

Week 3 weight: $4.9\ \text{lb}$

Week 4 weight: $5.4\ \text{lb}$

You have previously determined, using additional measurements, that the uncertainty in the average weight of the $5\text{-lb}$ bag is $\pm 0.4\ \text{lb}$. What is the average weight and percent uncertainty of the bag's weight, given these 4 measurements?

### Strategy

The average of the four measured weights is 5.1 lb. Observe that the measured average agrees with the nominal weight of 5 lb within the uncertainty of 0.4 lb. The uncertainty in this value, $\delta A$, is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$\%\ \text{unc} = \frac{\delta A}{A} \times 100\%. \qquad (1.8)$$

## Solution

Plug the known values into the equation:

$$\% \text{ unc} = \frac{0.4 \text{ lb}}{5.1 \text{ lb}} \times 100\% = 8\%. \tag{1.9}$$

## Discussion

We can conclude that the weight of the apple bag is $5 \text{ lb} \pm 8\%$. Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

# Worksheet

Mindfulness

# Worksheet

Mindfulness

Discussion and Solutions

# APPENDIX A Powers of Ten and Scientific Notation

In science, very large and very small decimal numbers are conveniently expressed in terms of powers of ten, some of which are listed below:

$$10^3 = 10 \times 10 \times 10 = 1000 \qquad 10^{-3} = \frac{1}{10 \times 10 \times 10} = 0.001$$

$$10^2 = 10 \times 10 = 100 \qquad 10^{-2} = \frac{1}{10 \times 10} = 0.01$$

$$10^1 = 10 \qquad 10^{-1} = \frac{1}{10} = 0.1$$

$$10^0 = 1$$

Using powers of ten, we can write the radius of the earth in the following way, for example:

$$\text{Earth radius} = 6\,380\,000 \text{ m} = 6.38 \times 10^6 \text{ m}$$

The factor of ten raised to the sixth power is ten multiplied by itself six times, or one million, so the earth's radius is 6.38 million meters. Alternatively, the factor of ten raised to the sixth power indicates that the decimal point in the term 6.38 is to be moved six places *to the right* to obtain the radius as a number without powers of ten.

For numbers less than one, negative powers of ten are used. For instance, the Bohr radius of the hydrogen atom is

$$\text{Bohr radius} = 0.000\,000\,000\,0529 \text{ m} = 5.29 \times 10^{-11} \text{ m}$$

The factor of ten raised to the minus eleventh power indicates that the decimal point in the term 5.29 is to be moved eleven places *to the left* to obtain the radius as a number without powers of ten. Numbers expressed with the aid of powers of ten are said to be in ***scientific notation.***

Calculations that involve the multiplication and division of powers of ten are carried out as in the following examples:

$$(2.0 \times 10^6)(3.5 \times 10^3) = (2.0 \times 3.5) \times 10^{6+3} = 7.0 \times 10^9$$

$$\frac{9.0 \times 10^7}{2.0 \times 10^4} = \left(\frac{9.0}{2.0}\right) \times 10^7 \times 10^{-4} = \left(\frac{9.0}{2.0}\right) \times 10^{7-4} = 4.5 \times 10^3$$

The general rules for such calculations are

$$\frac{1}{10^n} = 10^{-n} \qquad \text{(A-1)}$$

$$10^n \times 10^m = 10^{n+m} \quad \text{(Exponents added)} \qquad \text{(A-2)}$$

$$\frac{10^n}{10^m} = 10^{n-m} \quad \text{(Exponents subtracted)} \qquad \text{(A-3)}$$

where $n$ and $m$ are any positive or negative number.

Scientific notation is convenient because of the ease with which it can be used in calculations. Moreover, scientific notation provides a convenient way to express the significant figures in a number, as Appendix B discusses.

# APPENDIX B Significant Figures

The number of ***significant figures*** in a number is the number of digits whose values are known with certainty. For instance, a person's height is measured to be 1.78 m, with the measurement error being in the third decimal place. All three digits are known with certainty, so that the number contains three significant figures. If a zero is given as the last digit to the right of the decimal point, the zero is presumed to be significant. Thus, the number 1.780 m contains four significant figures. As another example, consider a distance of 1500 m. This number contains only two significant figures, the one and the five. The zeros immediately to the left of the unexpressed decimal point are not counted as significant figures. However, zeros located between significant figures are significant, so a distance of 1502 m contains four significant figures.

Scientific notation is particularly convenient from the point of view of significant figures. Suppose it is known that a certain distance is fifteen hundred meters, to four significant figures. Writing the number as 1500 m presents a problem because it implies that only two significant figures are known. In contrast, the scientific notation of $1.500 \times 10^3$ m has the advantage of indicating that the distance is known to four significant figures.

When two or more numbers are used in a calculation, the number of significant figures in the answer is limited by the number of significant figures in the original data. For instance, a rectangular garden with sides of 9.8 m and 17.1 m has an area of (9.8 m)(17.1 m). A calculator gives 167.58 $m^2$ for this product. However, one of the original lengths is known only to two significant figures, so the final answer is limited to only two significant figures and should be rounded off to 170 $m^2$. In general, ***when numbers are multiplied or divided, the number of significant figures in the final answer equals the smallest number of significant figures in any of the original factors.***

The number of significant figures in the answer to an addition or a subtraction is also limited by the original data. Consider the total distance along a biker's trail that consists of three segments with the distances shown as follows:

| | |
|---|---|
| | 2.5 km |
| | 11 km |
| | 5.26 km |
| Total | 18.76 km |

The distance of 11 km contains no significant figures to the right of the decimal point. Therefore, neither does the sum of the three distances, and the total distance should not be reported as 18.76 km. Instead, the answer is rounded off to 19 km. In general, ***when numbers are added or subtracted, the last significant figure in the answer occurs in the last column (counting from left to right) containing a number that results from a combination of digits that are all significant.*** In the answer of 18.76 km, the eight is the sum of 2 + 1 + 5, each digit being significant. However, the seven is the sum of 5 + 0 + 2, and the zero is not significant, since it comes from the 11-km distance, which contains no significant figures to the right of the decimal point.

Exercise 1: Standard Decimal Addition

**Problem:** A chemist mixes three liquid samples together. Their volumes are measured as 14.25 mL, 7.3 mL, and 0.115 mL. What is the total volume of the mixture reported to the correct number of significant figures?

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**Solution:** 1. **Align and Add:** Write out the standard mathematical sum:

$$\begin{array}{rrl} & 14.25 & \text{mL} \\ & 7.3 & \text{mL} \\ + & 0.115 & \text{mL} \\ \hline & 21.665 & \text{mL} \end{array}$$

2. **Identify the Limiting Column:** * 14.25 ends in the hundredths place (2 decimal places).

* 7.3 ends in the **tenths place** (1 decimal place). * 0.115 ends in the thousandths place (3 decimal places).

3. **Round the Answer:** Because 7.3 is only known to the tenths place, the final answer must be rounded to the tenths place. * 21.665 mL rounds to **21.7 mL**.

Exercise 2: Addition with Ambiguous/Trailing Zeros

**Problem:** An engineer is calculating the total length of a pipeline consisting of two sections. The first section is roughly measured to be 1300 m (where the trailing zeros are not significant). The second section is precisely measured to be 255 m. What is the total length?

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**Solution:** 1. **Align and Add:**

$$\begin{array}{rl} 1300 & \text{m} \\ +\ \ 255 & \text{m} \\ \hline 1555 & \text{m} \end{array}$$

2. **Identify the Limiting Column:** * Following the procedures learned (like the 1500 m example), the number 1300 m has only two significant figures (1 and 3). This means its last significant digit is in the **hundreds place**.

* 255 m has its last significant digit in the ones place.

3. **Round the Answer:** The final sum is limited by the hundreds place. * 1555 m rounded to the nearest hundred is **1600 m** (or $1.6 \times 10^3$ m in scientific notation).

Exercise 3: Addition Involving a Radical (Square Root)

**Problem:** Find the total perimeter length of a custom triangular bracket component by adding the calculated side length $\sqrt{16.0 \text{ cm}^2}$ to an adjacent side length of 3.45 cm.

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**Problem:** Find the total perimeter length of a custom triangular bracket component by adding the calculated side length $\sqrt{16.0 \text{ cm}^2}$ to an adjacent side length of 3.45 cm.

**Solution:** 1. **Step 1: Evaluate the Radical:** * When taking the square root of a value, the rule of thumb is that the result retains the **same number of significant figures** as the original value inside the radical.

* 16.0 $\text{cm}^2$ has **3 significant figures**. * $\sqrt{16.0} = 4.00$ cm (written with 3 significant figures to preserve accuracy to the hundredths place).

2. **Step 2: Perform the Addition:**

$$\begin{array}{rl} 4.00 & \text{cm} \\ +\ 3.45 & \text{cm} \\ \hline 7.45 & \text{cm} \end{array}$$

3. **Identify the Limiting Column:** * Both 4.00 cm and 3.45 cm extend precisely to the hundredths place (2 decimal places). * Therefore, no additional rounding is needed. The final answer is **7.45 cm**.

## Exercise 1: Density Calculation (Division)

**Problem:**

An object has a measured mass of 45.20 g and a volume of 15.6 mL. Calculate its density using the formula:

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

Report the answer with the correct number of significant figures based on the guidelines.

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Report the answer with the correct number of significant figures based on the guidelines.

**Solution:**

1. **Identify Significant Figures:**
   - Mass = 45.20 g has 4 significant figures (the trailing zero after the decimal point is significant).
   - Volume = 15.6 mL has 3 significant figures.
2. **Perform the Calculation:**
$$\text{Density} = \frac{45.20 \text{ g}}{15.6 \text{ mL}} \approx 2.89743589 \text{ g/mL}$$
3. **Apply the Rule:** According to the rule for multiplication and division, the final answer is limited by the factor with the smallest number of significant figures. Here, the volume (15.6 mL) has the fewest (3 significant figures).
4. **Round the Final Answer:** Rounding 2.89743589 . . . to 3 significant figures yields:
$$\mathbf{2.90} \text{ g/mL}$$

A chemical solution is prepared, and its final concentration needs to be determined using the following formula:

$$\text{Concentration} = \frac{\text{Mass of solute} \times \text{Purity factor}}{\text{Volume of solution}}$$

Given that the mass of the solute is 3.50 g, the purity factor is 0.98, and the volume of the solution is 0.2500 L, calculate the concentration to the correct number of significant figures.

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**Solution:**

1. **Identify Significant Figures:**

   - Mass of solute = 3.50 g has 3 significant figures.
   - Purity factor = 0.98 has 2 significant figures (leading zeros are not significant).
   - Volume = 0.2500 L has 4 significant figures (trailing zeros after a decimal point are significant).

2. **Perform the Calculation:**

$$\text{Concentration} = \frac{3.50 \times 0.98}{0.2500} = \frac{3.43}{0.2500} = 13.72 \text{ g/L}$$

3. **Apply the Rule:** The rule for multiplication and division states that the final result cannot have more significant figures than the least precise factor. The purity factor (0.98) has the fewest significant figures (2).

4. **Round the Final Answer:** Rounding 13.72 to 2 significant figures yields:

$$\mathbf{14} \text{ g/L}$$

**Problem:**

The final velocity $v$ of an object dropping from rest over a distance $d$ is determined by the formula:

$$v = \sqrt{2gd}$$

Calculate the final velocity if the acceleration due to gravity is $g = 9.81\ \mathrm{m/s^2}$ and the distance is $d = 12.50$ m. (Note: The number 2 is an exact mathematical constant and does not limit the significant figures).

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**Solution:**

1. **Identify Significant Figures:**
   - Acceleration ($g$) = $9.81 \text{ m/s}^2$ has 3 significant figures.
   - Distance ($d$) = 12.50 m has 4 significant figures.
   - The constant 2 is exact and has an infinite number of significant figures.

2. **Perform the Calculation:** First, compute the product inside the radical:

$$2 \times 9.81 \times 12.50 = 245.25$$

   Next, extract the square root:

$$v = \sqrt{245.25} \approx 15.6604598 \text{ m/s}$$

3. **Apply the Rule:** For operations containing roots or radicals, the result inherits the precision limitations of the values inside. Since the core multiplication inside is constrained by $g$ (3 significant figures), the value under the root effectively carries 3 significant figures of precision, meaning the final answer must be rounded to 3 significant figures.

4. **Round the Final Answer:** Rounding $15.6604598\ldots$ to 3 significant figures yields:

$$\mathbf{15.7} \text{ m/s}$$

# Approximation.

### EXAMPLE 1.3

## Approximate the Height of a Building

Can you approximate the height of one of the buildings on your campus, or in your neighborhood? Let us make an approximation based upon the height of a person. In this example, we will calculate the height of a 39-story building.

### Strategy

Think about the average height of an adult male. We can approximate the height of the building by scaling up from the height of a person.

### Solution

Based on information in the example, we know there are 39 stories in the building. If we use the fact that the height of one story is approximately equal to about the length of two adult humans (each human is about 2-m tall), then we can estimate the total height of the building to be

$$\frac{2\ \text{m}}{1\ \text{person}} \times \frac{2\ \text{person}}{1\ \text{story}} \times 39\ \text{stories} = 156\ \text{m}. \tag{1.13}$$

### Discussion

You can use known quantities to determine an approximate measurement of unknown quantities. If your hand measures 10 cm across, how many hand lengths equal the width of your desk? What other measurements can you approximate besides length?

# Approximation

The U.S. federal debt in 2021 was a little more than $28 trillion. Most of us do not have any concept of how much even one trillion actually is. Suppose that you were given a trillion dollars in $100 bills. If you made 100-bill stacks and used them to evenly cover a football field (between the end zones), make an approximation of how high the money pile would become. (We will use feet/inches rather than meters here because football fields are measured in yards.) One of your friends says 3 in., while another says 10 ft. What do you think?

### Strategy

When you imagine the situation, you probably envision thousands of small stacks of 100 wrapped $100 bills, such as you might see in movies or at a bank. Since this is an easy-to-approximate quantity, let us start there. We can find the volume of a stack of 100 bills, find out how many stacks make up one trillion dollars, and then set this volume equal to the area of the football field multiplied by the unknown height.

**Solution**

(1) Calculate the volume of a stack of 100 bills. The dimensions of a single bill are approximately 3 in. by 6 in. A stack of 100 of these is about 0.5 in. thick. So the total volume of a stack of 100 bills is:

$$\begin{aligned} \text{volume of stack} &= \text{length} \times \text{width} \times \text{height}, \\ \text{volume of stack} &= 6\text{ in.} \times 3\text{ in.} \times 0.5\text{ in.}, \\ \text{volume of stack} &= 9\text{ in.}^3. \end{aligned} \tag{1.14}$$

(2) Calculate the number of stacks. Note that a trillion dollars is equal to $\$1 \times 10^{12}$, and a stack of one-hundred $\$100$ bills is equal to $\$10{,}000$, or $\$1 \times 10^4$. The number of stacks you will have is:

$$\$1 \times 10^{12}(\text{a trillion dollars})/\ \$1 \times 10^4 \text{ per stack} = 1 \times 10^8 \text{ stacks.} \tag{1.15}$$

(3) Calculate the area of a football field in square inches. The area of a football field is $100 \text{ yd} \times 50 \text{ yd}$, which gives $5{,}000 \text{ yd}^2$. Because we are working in inches, we need to convert square yards to square inches:

$$\text{Area} = 5{,}000 \text{ yd}^2 \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in.}}{1 \text{ ft}} \times \frac{12 \text{ in.}}{1 \text{ ft}} = 6{,}480{,}000 \text{ in.}^2,$$
$$\text{Area} \approx 6 \times 10^6 \text{ in.}^2. \tag{1.16}$$

This conversion gives us $6 \times 10^6 \text{ in.}^2$ for the area of the field. (Note that we are using only one significant figure in these calculations.)

(4) Calculate the total volume of the bills. The volume of all the $\$100$-bill stacks is $9 \text{ in.}^3/\text{stack} \times 10^8 \text{ stacks} = 9 \times 10^8 \text{ in.}^3$.

(5) Calculate the height. To determine the height of the bills, use the equation:

$$\begin{aligned} \text{volume of bills} &= \text{area of field} \times \text{height of money:} \\ \text{Height of money} &= \frac{\text{volume of bills}}{\text{area of field}}, \\ \text{Height of money} &= \frac{9 \times 10^8 \text{in.}^3}{6 \times 10^6 \text{in.}^2} = 1.33 \times 10^2 \text{in.}, \\ \text{Height of money} &\approx 1 \times 10^2 \text{in.} = 100 \text{ in.} \end{aligned} \tag{1.17}$$

The height of the money will be about 100 in. high. Converting this value to feet gives

$$100 \text{ in.} \times \frac{1 \text{ ft}}{12 \text{ in.}} = 8.33 \text{ ft} \approx 8 \text{ ft}. \tag{1.18}$$

### Discussion

The final approximate value is much higher than the early estimate of 3 in., but the other early estimate of 10 ft (120 in.) was roughly correct. How did the approximation measure up to your first guess? What can this exercise tell you in terms of rough “guesstimates” versus carefully calculated approximations?

## Check Your Understanding

Using mental math and your understanding of fundamental units, approximate the area of a regulation basketball court. Describe the process you used to arrive at your final approximation.

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### Solution

An average male is about two meters tall. It would take approximately 15 males laid out end to end to cover the length, and about 7 to cover the width. That gives an approximate area of $420\ \mathrm{m}^2$.

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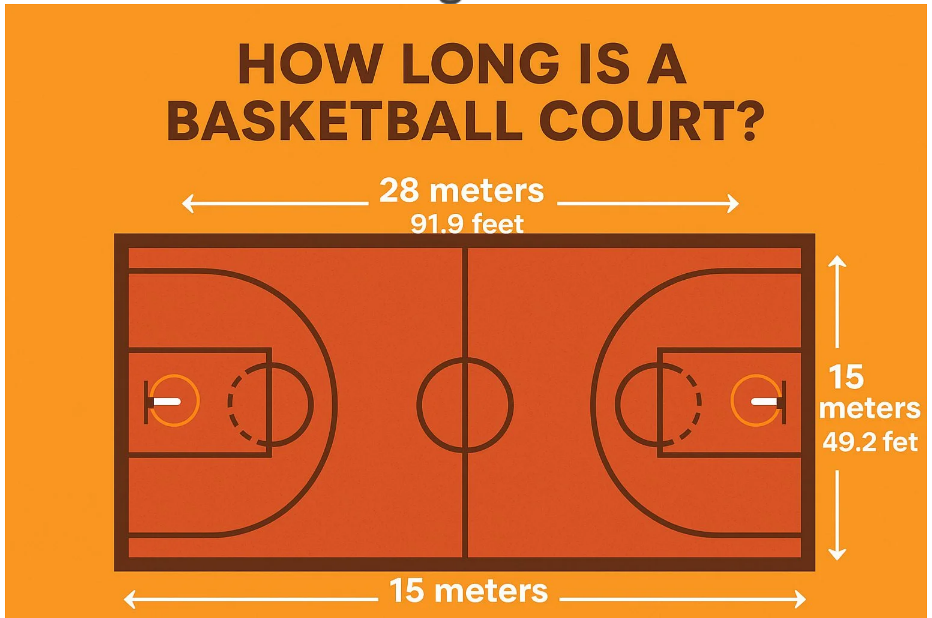

# Chapter 2

# Kinematics in One Dimension

## 2.1 Displacement

There are two aspects to any motion. In a purely descriptive sense, there is the movement itself. Is it rapid or slow, for instance? Then, there is the issue of what causes the motion or what changes it, which requires that forces be considered. ***Kinematics*** deals with the concepts that are needed to describe motion, without any reference to forces. The present chapter discusses these concepts as they apply to motion in one dimension, and the next chapter treats two-dimensional motion. ***Dynamics*** deals with the effect that forces have on motion, a topic that is considered in Chapter 4. Together, kinematics and dynamics form the branch of physics known as ***mechanics.*** We turn now to the first of the kinematics concepts to be discussed, which is displacement.

To describe the motion of an object, we must be able to specify the location of the object at all times, and Figure 2.1 shows how to do this for one-dimensional motion. In this drawing, the initial position of a car is indicated by the vector labeled $\vec{\mathbf{x}}_0$. The length of $\vec{\mathbf{x}}_0$ is the distance of the car from an arbitrarily chosen origin. At a later time the car has moved to a new position, which is indicated by the vector $\vec{\mathbf{x}}$. The ***displacement*** of the car $\Delta\vec{\mathbf{x}}$ (read as "delta x" or "the change in x") is a vector drawn from the initial position to the final position. Displacement is a vector quantity in the sense discussed in Section 1.5, for it conveys both a magnitude (the distance between the initial and final positions) and a direction. The displacement can be related to $\vec{\mathbf{x}}_0$ and $\vec{\mathbf{x}}$ by noting from the drawing that

$$\vec{\mathbf{x}}_0 + \Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} \quad \text{or} \quad \Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$$

Thus, the displacement $\Delta\vec{\mathbf{x}}$ is the difference between $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}_0$, and the Greek letter delta ($\Delta$) is used to signify this difference. It is important to note that the change in any variable is always the final value minus the initial value.

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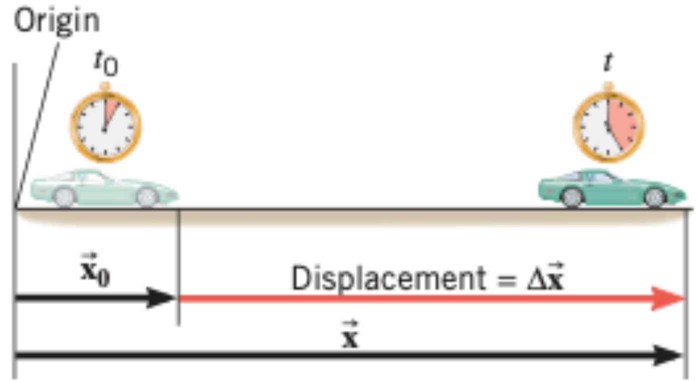


**Figure 2.1** The displacement $\Delta\vec{\mathbf{x}}$ is a vector that points from the initial position $\vec{\mathbf{x}}_0$ to the final position $\vec{\mathbf{x}}$.

### Definition of Displacement

The displacement is a vector that points from an object's initial position to its final position and has a magnitude that equals the shortest distance between the two positions.

***SI Unit of Displacement:*** meter (m)

## 2.1 Displacement

# Chapter 2

**LEARNING OBJECTIVES**

By the end of this section, you will be able to:

- Define position, displacement, distance, and distance traveled.
- Explain the relationship between position and displacement.
- Distinguish between displacement and distance traveled.
- Calculate displacement and distance given initial position, final position, and the path between the two.

# Chapter 2

## Position

In order to describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See Figure 2.3.) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See Figure 2.4.)

# Chapter 2

## Displacement

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement**. The word "displacement" implies that an object has moved, or has been displaced.

**DISPLACEMENT**

Displacement is the *change in position* of an object:

$$\Delta x = x_\mathrm{f} - x_0, \tag{2.1}$$

where $\Delta x$ is displacement, $x_\mathrm{f}$ is the final position, and $x_0$ is the initial position.

Suppose that the initial position of the car in Figure 2.1 is $\vec{\mathbf{x}}_0$ when the time is $t_0$. A little later that car arrives at the final position $\vec{\mathbf{x}}$ at the time $t$. The difference between these times is the time required for the car to travel between the two positions. We denote this difference by the shorthand notation $\Delta t$ (read as "delta $t$"), where $\Delta t$ represents the final time $t$ minus the initial time $t_0$:

$$\Delta t = \underbrace{t - t_0}_{\text{Elapsed time}}$$

Note that $\Delta t$ is defined in a manner analogous to $\Delta\vec{\mathbf{x}}$, which is the final position minus the initial position ($\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$). Dividing the displacement $\Delta\vec{\mathbf{x}}$ of the car by the elapsed time $\Delta t$ gives the ***average velocity*** of the car. It is customary to denote the average value of a quantity by placing a horizontal bar above the symbol representing the quantity. The average velocity, then, is written as $\overline{\vec{\mathbf{v}}}$, as specified in Equation 2.2:

Cha

## MISCONCEPTION ALERT: DISTANCE TRAVELED VS. MAGNITUDE OF DISPLACEMENT

It is important to note that the *distance traveled*, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we nearly always deal with displacement and magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

# Chapter 2

**Check Your Understanding**

A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is their displacement? (b) What distance do they ride? (c) What is the magnitude of their displacement?

# Chap

- We may assume the direction ‘east’ is our positive direction, like in a typical x axis in a Cartesian plane.

Chap

## Check Your Understanding

A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is their displacement? (b) What distance do they ride? (c) What is the magnitude of their displacement?

[Show/Hide Solution]

### Solution

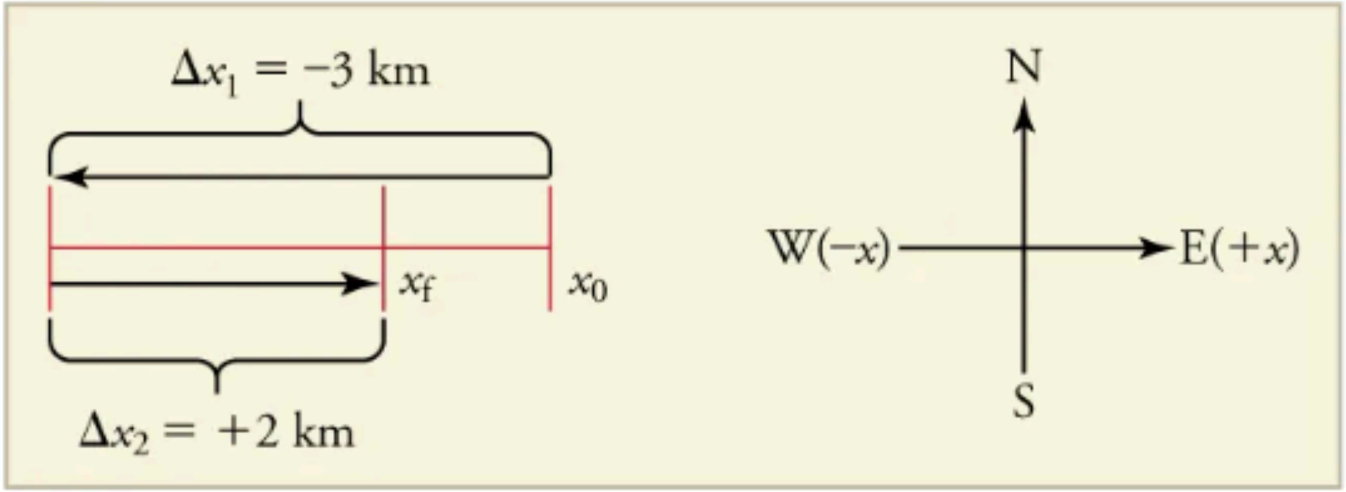


Figure 2.5

(a) The rider's displacement is $\Delta x = x_f - x_0 = -1$ km. (The displacement is negative because we take east to be positive and west to be negative.)

(b) The distance traveled is $3$ km $+ 2$ km $= 5$ km.

(c) The magnitude of the displacement is $1$ km.

# Chapter 2

## Time

As discussed in Physical Quantities and Units, the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple—**time** is *change*, or the interval over which change occurs. It is impossible to know that time has passed unless something changes.

The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. **Elapsed time** $\Delta t$ is the difference between the ending time and beginning time,

$$\Delta t = t_{\mathrm{f}} - t_0, \tag{2.4}$$

where $\Delta t$ is the change in time or elapsed time, $t_{\mathrm{f}}$ is the time at the end of the motion, and $t_0$ is the time at the beginning of the motion. (As usual, the delta symbol, $\Delta$, means the change in the quantity that follows it.)

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where $\Delta t$ is the change in time or elapsed time, $t_{\mathrm{f}}$ is the time at the end of the motion, and $t_0$ is the time at the beginning of the motion. (As usual, the delta symbol, $\Delta$, means the change in the quantity that follows it.)

Life is simpler if the beginning time $t_0$ is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If $t_0 = 0$, then $\Delta t = t_{\mathrm{f}} \equiv t$.

In this text, for simplicity's sake,

- motion starts at time equal to zero $(t_0 = 0)$
- the symbol $t$ is used for elapsed time unless otherwise specified $(\Delta t = t_{\mathrm{f}} \equiv t)$

## 2.2 Speed and Velocity

### Average Speed

One of the most obvious features of an object in motion is how fast it is moving. If a car travels 200 meters in 10 seconds, we say its average speed is 20 meters per second, the ***average speed*** being the distance traveled divided by the time required to cover the distance:

$$\text{Average speed} = \frac{\text{Distance}}{\text{Elapsed time}} \tag{2.1}$$

Equation 2.1 indicates that the unit for average speed is the unit for distance divided by the unit for time, or meters per second (m/s) in SI units. Example 1 illustrates how the idea of average speed is used.

#### Example 1 Distance Run by a Jogger

How far does a jogger run in 1.5 hours (5400 s) if his average speed is 2.22 m/s?

***Reasoning*** The average speed of the jogger is the average distance per second that he travels. Thus, the distance covered by the jogger is equal to the average distance per second (his average speed) multiplied by the number of seconds (the elapsed time) that he runs.

***Solution*** To find the distance run, we rewrite Equation 2.1 as

$$\text{Distance} = (\text{Average speed})(\text{Elapsed time}) = (2.22\ \text{m/s})(5400\ \text{s}) = \boxed{12\,000\ \text{m}}$$

## Velocity

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

### AVERAGE VELOCITY

**Average velocity** is *displacement (change in position) divided by the time of travel*,

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}, \tag{2.5}$$

where $\bar{v}$ is the *average* (indicated by the bar over the $v$) velocity, $\Delta x$ is the change in position (or displacement), and $x_f$ and $x_0$ are the final and beginning positions at times $t_f$ and $t_0$, respectively. If the starting time $t_0$ is taken to be zero, then the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}. \tag{2.6}$$

$$\bar{v} = \frac{\Delta x}{t}. \tag{2.6}$$

Notice that this definition indicates that *velocity is a vector because displacement is a vector*. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move –4 m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}. \tag{2.7}$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

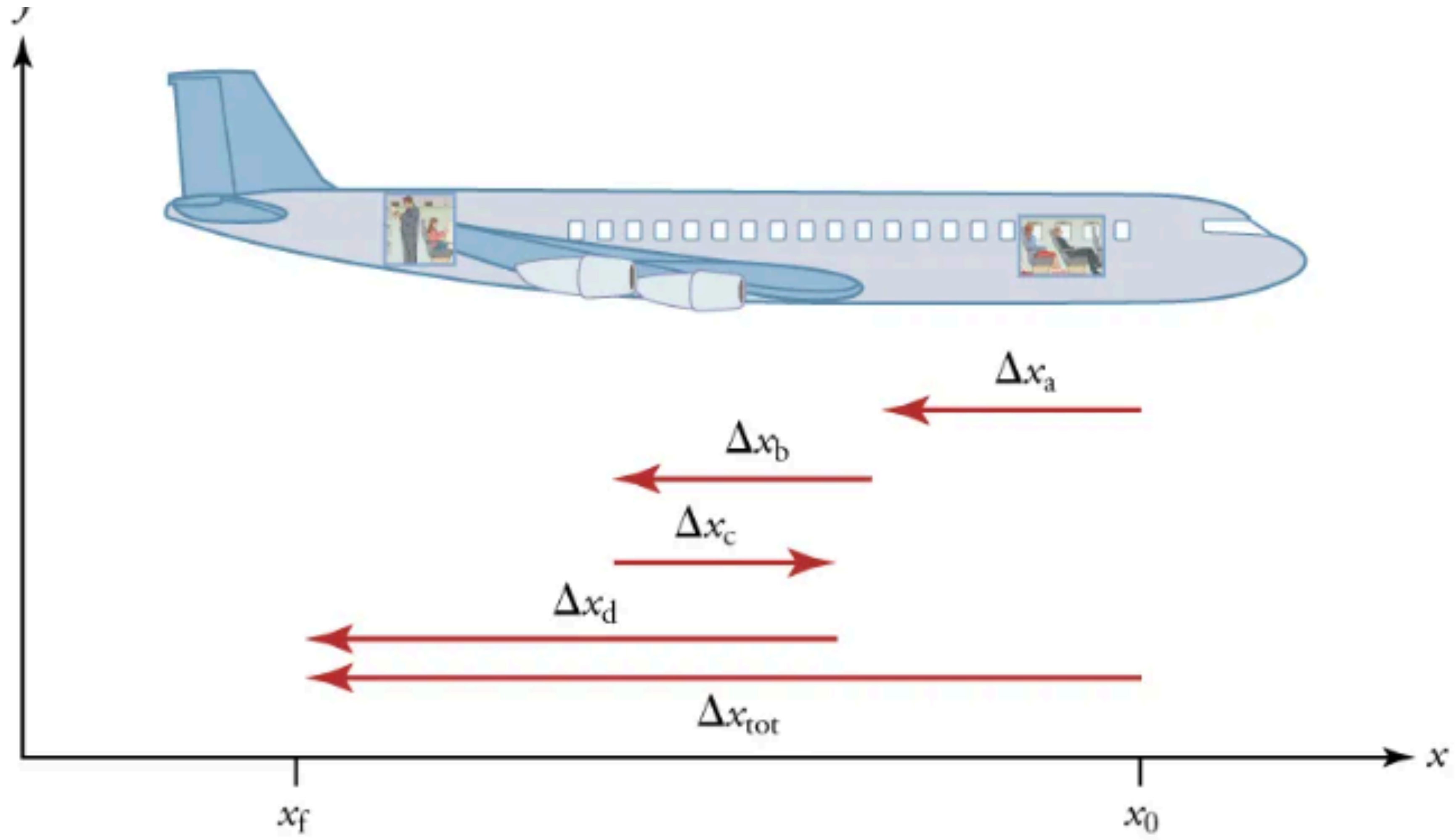



**Figure 2.9** A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the *instantaneous velocity* or the *velocity at a specific instant*. A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) **Instantaneous velocity** $v$ is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity, $v$, at a precise instant $t$ can involve taking a limit, a calculus operation beyond the scope of this text. However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

## Speed

In everyday language, most people use the terms “speed” and “velocity” interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus *speed is a scalar*. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

**Instantaneous speed** is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of –3.0 m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than the magnitude of displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car’s odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is *not* simply the magnitude of average velocity.

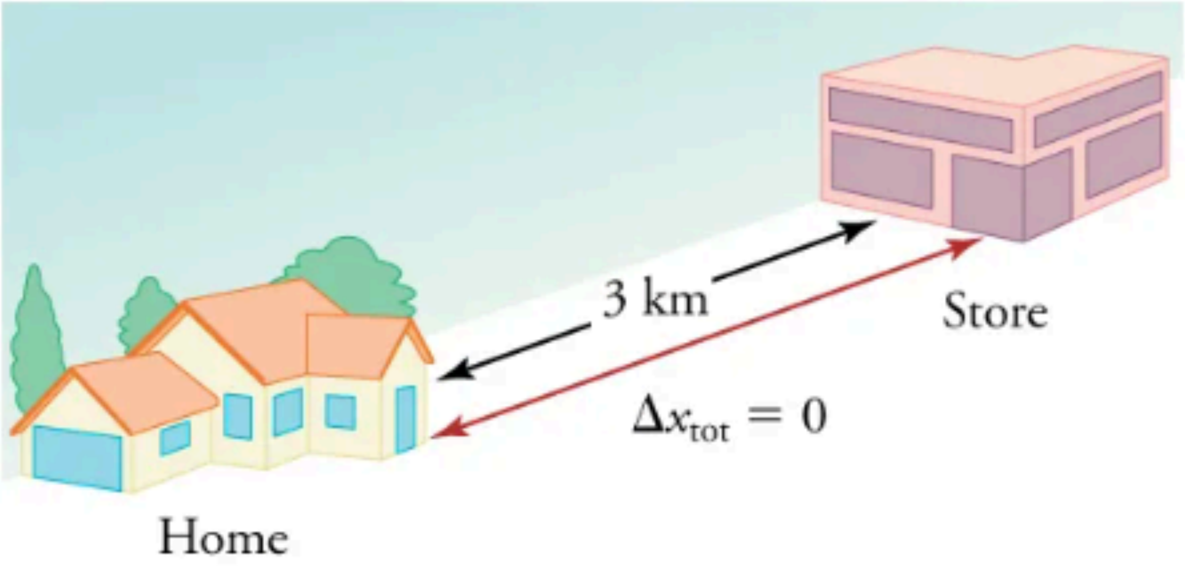
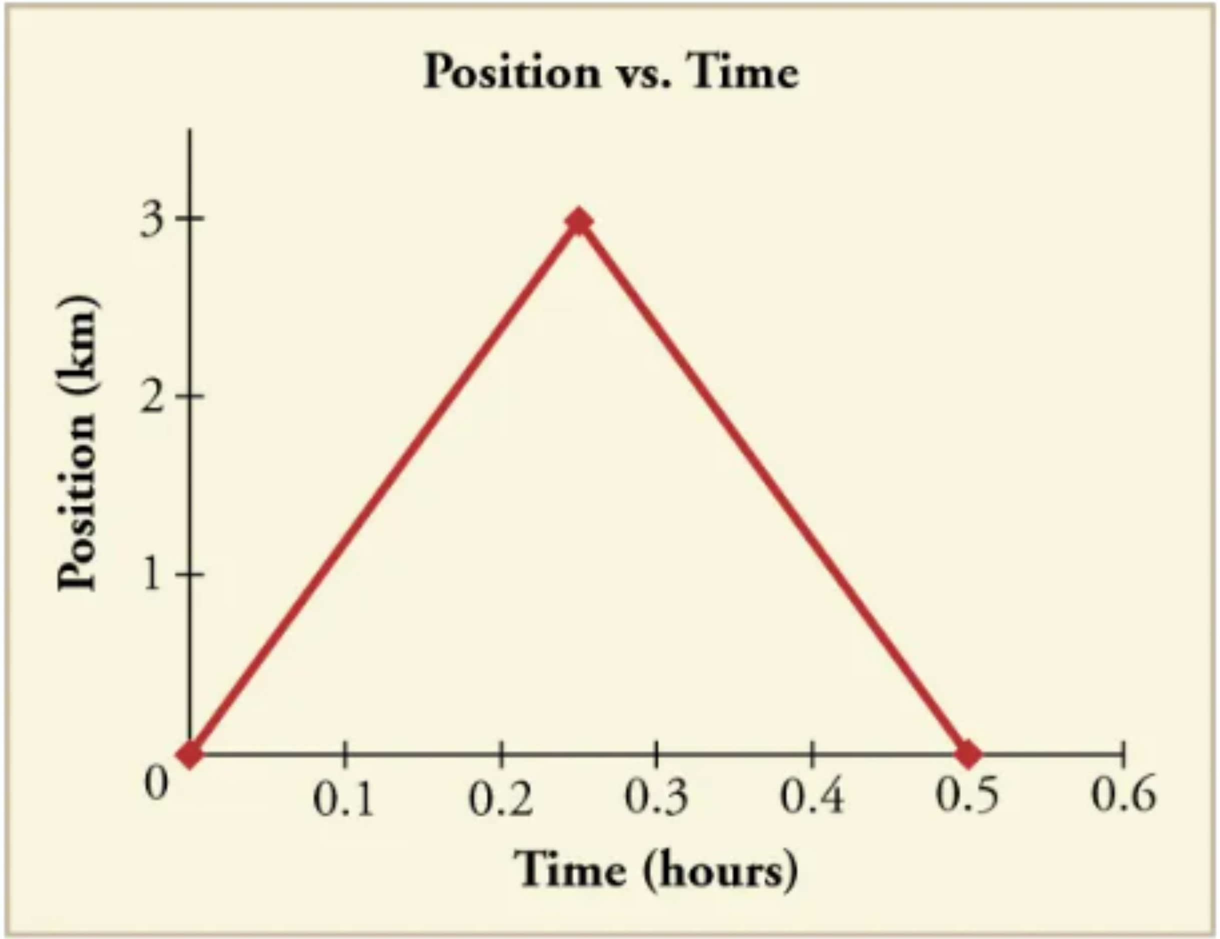



**Figure 2.10** During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in Figure 2.11. (Note that these graphs depict a very simplified **model** of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we'll probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)



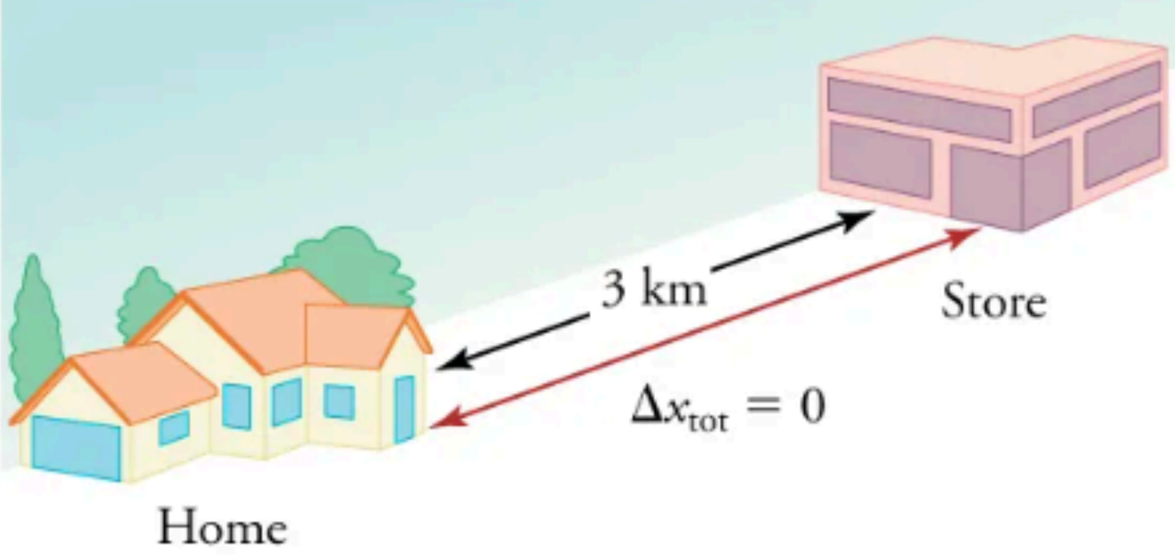
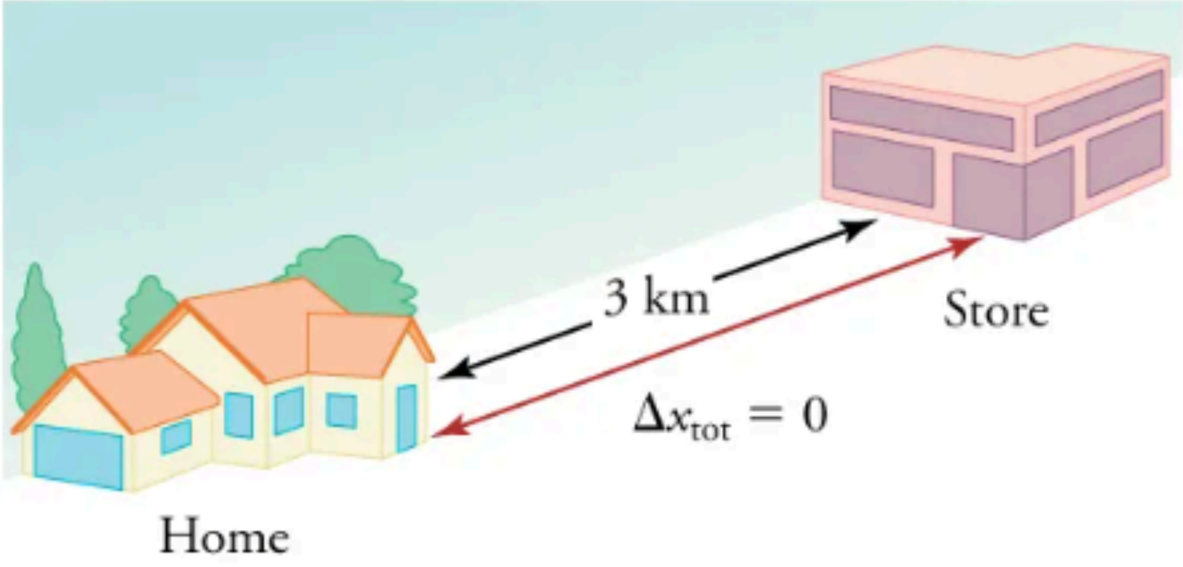




**Figure 2.10** During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

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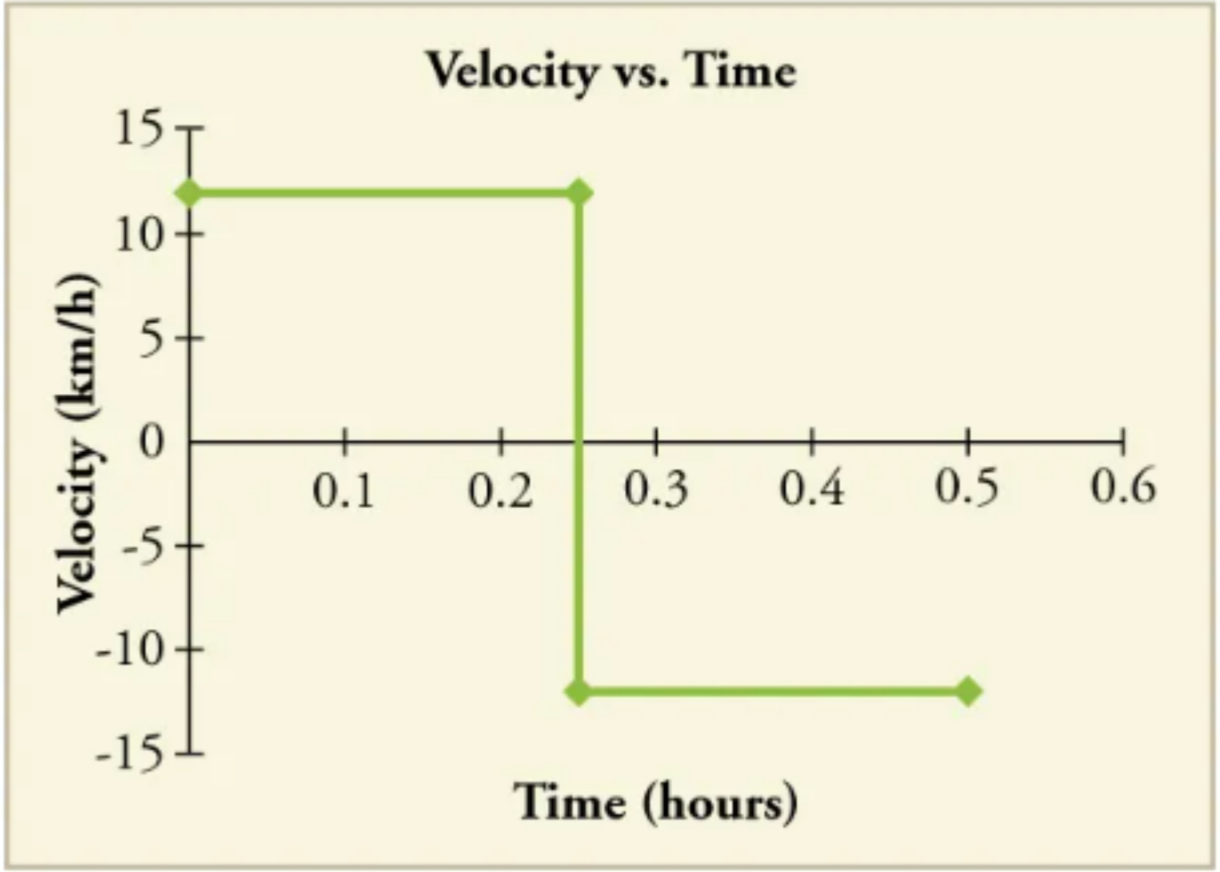
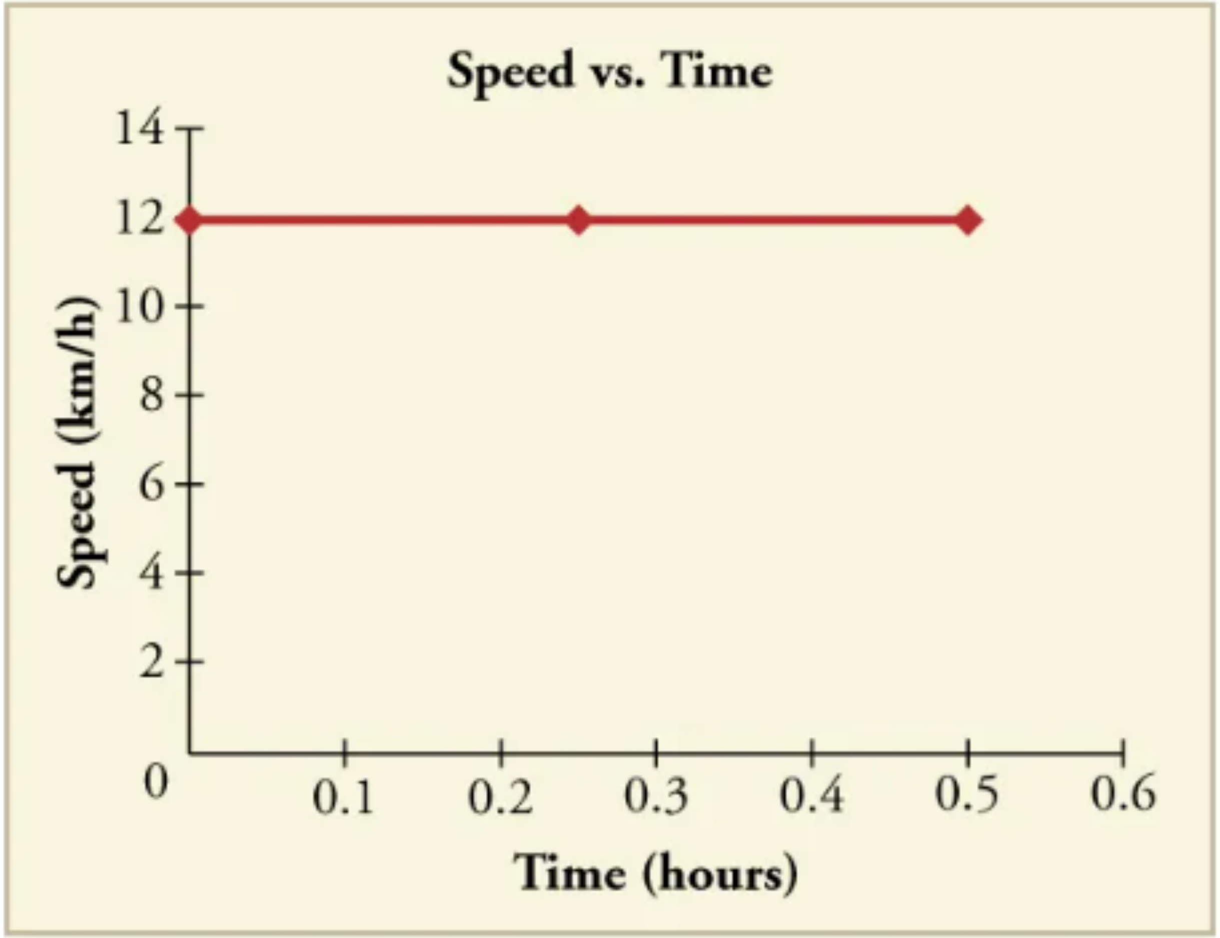



**Figure 2.11** Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.

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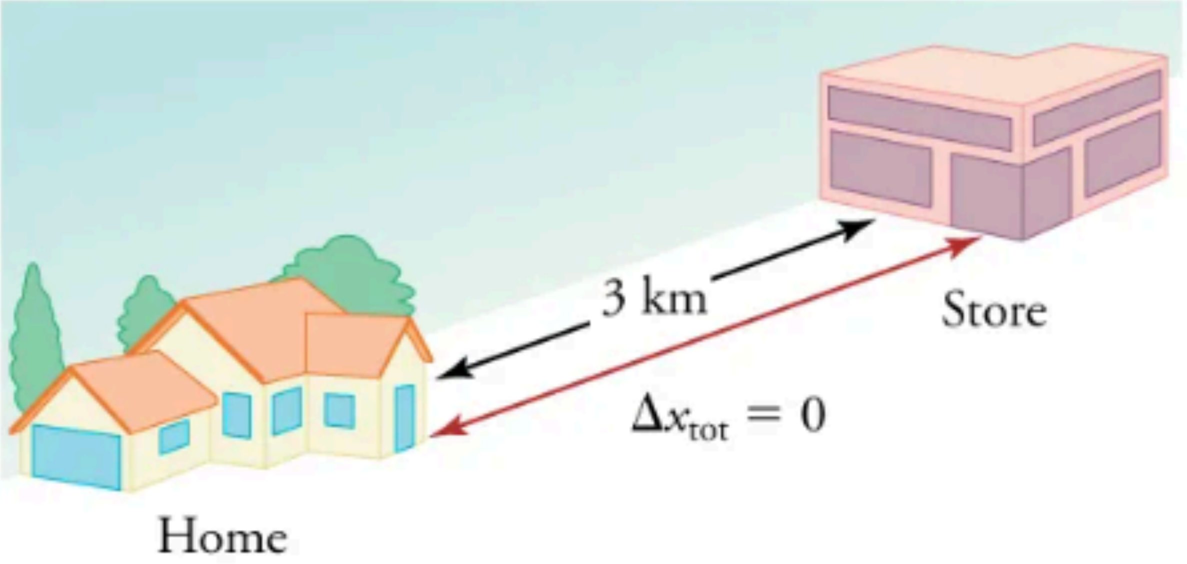


**Figure 2.10** During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Suppose that the initial position of the car in Figure 2.1 is $\vec{\mathbf{x}}_0$ when the time is $t_0$. A little later that car arrives at the final position $\vec{\mathbf{x}}$ at the time $t$. The difference between these times is the time required for the car to travel between the two positions. We denote this difference by the shorthand notation $\Delta t$ (read as "delta $t$"), where $\Delta t$ represents the final time $t$ minus the initial time $t_0$:

$$\Delta t = \underbrace{t - t_0}_{\text{Elapsed time}}$$

Note that $\Delta t$ is defined in a manner analogous to $\Delta\vec{\mathbf{x}}$, which is the final position minus the initial position ($\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$). Dividing the displacement $\Delta\vec{\mathbf{x}}$ of the car by the elapsed time $\Delta t$ gives the ***average velocity*** of the car. It is customary to denote the average value of a quantity by placing a horizontal bar above the symbol representing the quantity. The average velocity, then, is written as $\overline{\vec{\mathbf{v}}}$, as specified in Equation 2.2:

### Definition of Average Velocity

$$\text{Average velocity} = \frac{\text{Displacement}}{\text{Elapsed time}}$$

$$\overline{\vec{\mathbf{v}}} = \frac{\vec{\mathbf{x}} - \vec{\mathbf{x}}_0}{t - t_0} = \frac{\Delta\vec{\mathbf{x}}}{\Delta t} \tag{2.2}$$

***SI Unit of Average Velocity:*** meter per second (m/s)

Average velocity is a vector that points in the same direction as the displacement in Equation 2.2. Figure 2.2 illustrates that the velocity of an object confined to move along a line can point either in one direction or in the opposite direction. As with displacement, we will use plus and minus signs to indicate the two possible directions. If the displacement points in the positive direction, the average velocity is positive. Conversely, if the displacement points in the negative direction, the average velocity is negative. Example 2 illustrates these features of average velocity.

**Figure 2.2** The boats in this photograph are traveling in opposite directions; in other words, the velocity of one boat points in a direction that is opposite to the velocity of the other boat. (© Michael Dietrick/ age fotostock)

$$\bar{v} = \frac{\Delta x}{\Delta t} \implies \Delta t = \frac{\Delta x}{\bar{v}}$$

## Example 2 The World's Fastest Jet-Engine Car

Andy Green in the car *ThrustSSC* set a world record of 341.1 m/s (763 mi/h) in 1997. The car was powered by two jet engines, and it was the first one officially to exceed the speed of sound. To establish such a record, the driver makes two runs through the course, one in each direction, to nullify wind effects. Figure 2.3*a* shows that the car first travels from left to right and covers a distance of 1609 m (1 mile) in a time of 4.740 s. Figure 2.3*b* shows that in the reverse direction, the car covers the same distance in 4.695 s. From these data, determine the average velocity for each run.

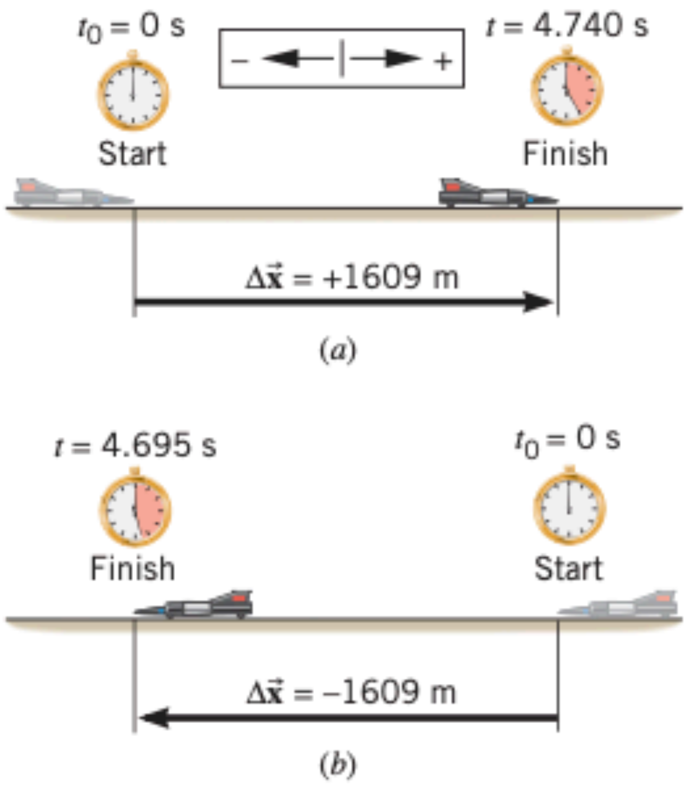


**Figure 2.3** The arrows in the box at the top of the drawing indicate the positive and negative directions for the displacements of the car, as explained in Example 2.

**Reasoning** Average velocity is defined as the displacement divided by the elapsed time. In using this definition we recognize that the displacement is not the same as the distance traveled. Displacement takes the direction of the motion into account, and distance does not. During both runs, the car covers the same distance of 1609 m. However, for the first run the displacement is $\Delta\vec{\mathbf{x}} = +1609$ m, while for the second it is $\Delta\vec{\mathbf{x}} = -1609$ m. The plus and minus signs are essential, because the first run is to the right, which is the positive direction, and the second run is in the opposite or negative direction.

**Solution** According to Equation 2.2, the average velocities are

*Run 1* $$\bar{\vec{\mathbf{v}}} = \frac{\Delta\vec{\mathbf{x}}}{\Delta t} = \frac{+1609 \text{ m}}{4.740 \text{ s}} = \boxed{+339.5 \text{ m/s}}$$

*Run 2* $$\bar{\vec{\mathbf{v}}} = \frac{\Delta\vec{\mathbf{x}}}{\Delta t} = \frac{-1609 \text{ m}}{4.695 \text{ s}} = \boxed{-342.7 \text{ m/s}}$$

## Instantaneous Velocity

Suppose the magnitude of your average velocity for a long trip was 20 m/s. This value, being an average, does not convey any information about how fast you were moving or the direction of the motion at any instant during the trip. Both can change from one instant to another. Surely there were times when your car traveled faster than 20 m/s and times when it traveled more slowly. The ***instantaneous velocity*** $\vec{\mathbf{v}}$ of the car indicates how fast the car moves and the direction of the motion at each instant of time. The magnitude of the instantaneous velocity is called the ***instantaneous speed,*** and it is the number (with units) indicated by the speedometer.

The instantaneous velocity at any point during a trip can be obtained by measuring the time interval $\Delta t$ for the car to travel a *very small* displacement $\Delta\vec{\mathbf{x}}$. We can then compute the average velocity over this interval. If the time $\Delta t$ is small enough, the instantaneous velocity does not change much during the measurement. Then, the instantaneous velocity $\vec{\mathbf{v}}$ at the point of interest is approximately equal to ($\approx$) the average velocity $\overline{\vec{\mathbf{v}}}$ computed over the interval, or $\vec{\mathbf{v}} \approx \overline{\vec{\mathbf{v}}} = \Delta\vec{\mathbf{x}}/\Delta t$ (for sufficiently small $\Delta t$). In fact, in the limit that $\Delta t$ becomes infinitesimally small, the instantaneous velocity and the average velocity become equal, so that

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{x}}}{\Delta t} \tag{2.3}$$

The notation $\lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{\mathbf{x}}}{\Delta t}$ means that the ratio $\Delta\vec{\mathbf{x}}/\Delta t$ is defined by a limiting process in which smaller and smaller values of $\Delta t$ are used, so small that they approach zero. As smaller values of $\Delta t$ are used, $\Delta\vec{\mathbf{x}}$ also becomes smaller. However, the ratio $\Delta\vec{\mathbf{x}}/\Delta t$ does *not* become zero but, rather, approaches the value of the instantaneous velocity. For brevity, we will use the word *velocity* to mean "instantaneous velocity" and *speed* to mean "instantaneous speed."

## Check Your Understanding

(*The answers are given at the end of the book*).

2. Is the average speed of a vehicle a vector or a scalar quantity?
3. Two buses depart from Chicago, one going to New York and one to San Francisco. Each bus travels at a speed of 30 m/s. Do they have equal velocities?
4. One of the following statements is incorrect. **(a)** The car traveled around the circular track at a constant velocity. **(b)** The car traveled around the circular track at a constant speed. Which statement is incorrect?
5. A straight track is 1600 m in length. A runner begins at the starting line, runs due east for the full length of the track, turns around and runs halfway back. The time for this run is five minutes. What is the runner's average velocity, and what is his average speed?
6. The average velocity for a trip has a positive value. Is it possible for the instantaneous velocity at a point during the trip to have a negative value?

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**CYU 2:** scalar quantity

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**CYU 2:** scalar quantity
**CYU 3:** No.

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4. One of the following statements is incorrect. **(a)** The car traveled around the circular track at a constant velocity. **(b)** The car traveled around the circular track at a constant speed. Which statement is incorrect?
5. A straight track is 1600 m in length. A runner begins at the starting line, runs due east for the full length of the track, turns around and runs halfway back. The time for this run is five minutes. What is the runner's average velocity, and what is his average speed?
6. The average velocity for a trip has a positive value. Is it possible for the instantaneous velocity at a point during the trip to have a negative value?

**CYU 2:** scalar quantity
**CYU 3:** No.
**CYU 4:** a

## Check Your Understanding

(*The answers are given at the end of the book*).

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**CYU 6:** Yes.

The meaning of ***average acceleration*** can be illustrated by considering a plane during takeoff. Figure 2.4 focuses attention on how the plane's velocity changes along the runway. During an elapsed time interval $\Delta t = t - t_0$, the velocity changes from an initial value of $\vec{\mathbf{v}}_0$ to a final velocity of $\vec{\mathbf{v}}$. The change $\Delta\vec{\mathbf{v}}$ in the plane's velocity is its final velocity minus its initial velocity, so that $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}} - \vec{\mathbf{v}}_0$. The average acceleration $\overline{\vec{\mathbf{a}}}$ is defined in the following manner, to provide a measure of how much the velocity changes per unit of elapsed time.

### Definition of Average Acceleration

$$\text{Average acceleration} = \frac{\text{Change in velocity}}{\text{Elapsed time}}$$

$$\overline{\vec{\mathbf{a}}} = \frac{\vec{\mathbf{v}} - \vec{\mathbf{v}}_0}{t - t_0} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} \tag{2.4}$$

***SI Unit of Average Acceleration:*** meter per second squared ($m/s^2$)

The average acceleration $\overline{\vec{\mathbf{a}}}$ is a vector that points in the same direction as $\Delta\vec{\mathbf{v}}$, the change in the velocity. Following the usual custom, plus and minus signs indicate the two possible directions for the acceleration vector when the motion is along a straight line.

We are often interested in an object's acceleration at a particular instant of time. The ***instantaneous acceleration*** $\vec{\mathbf{a}}$ can be defined by analogy with the procedure used in Section 2.2 for instantaneous velocity:

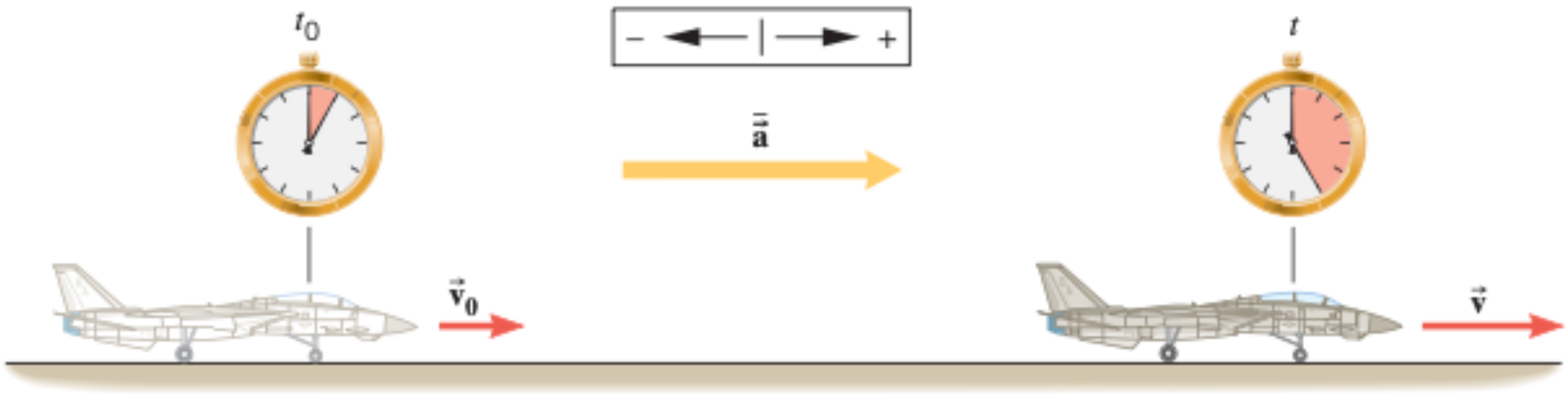
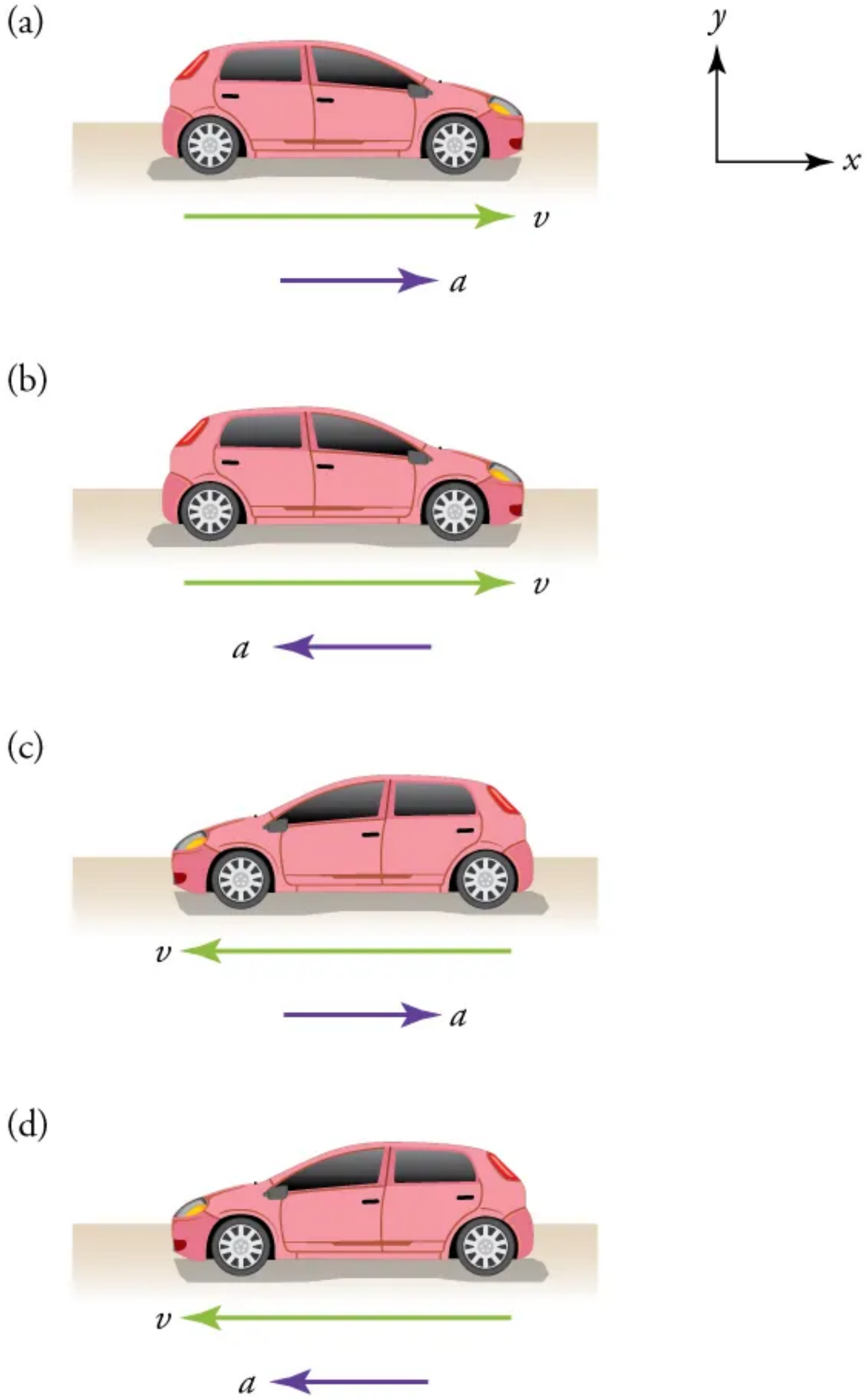



**Figure 2.4** During takeoff, the plane accelerates from an initial velocity $\vec{\mathbf{v}}_0$ to a final velocity $\vec{\mathbf{v}}$ during the time interval $\Delta t = t - t_0$.




- Figure 2.14 (a) This car is speeding up as it moves toward the right. It therefore has positive acceleration in our coordinate system. (b) This car is slowing down as it moves toward the right. Therefore, it has negative acceleration in our coordinate system, because its acceleration is toward the left. The car is also decelerating: the direction of its acceleration is opposite to its direction of motion. (c) This car is moving toward the left, but slowing down over time. Therefore, its acceleration is positive in our coordinate system because it is toward the right. However, the car is decelerating because its acceleration is opposite to its motion. (d) This car is speeding up as it moves toward the left. It has negative acceleration because it is accelerating toward the left. However, because its acceleration is in the same direction as its motion, it is speeding up (not decelerating).

## Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

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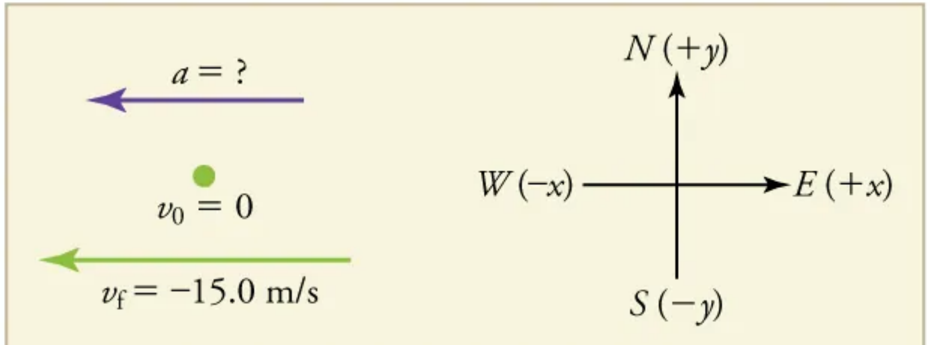
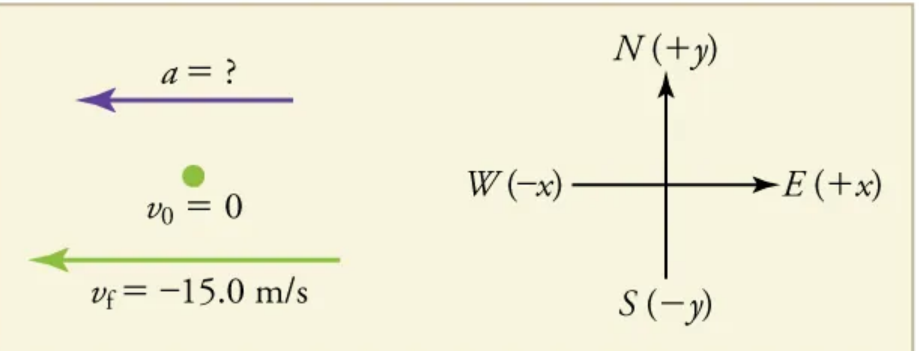


## Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

We can solve this problem by identifying $\Delta v$ and $\Delta t$ from the given information and then calculating the average acceleration directly from the equation



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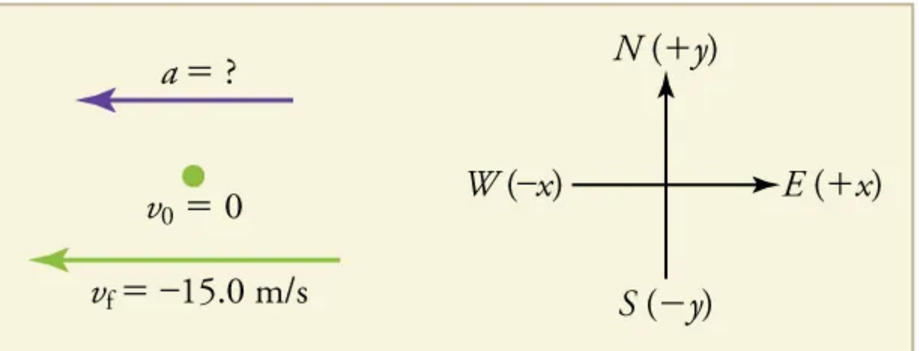
We can solve this problem by identifying $\Delta v$ and $\Delta t$ from the given information and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.



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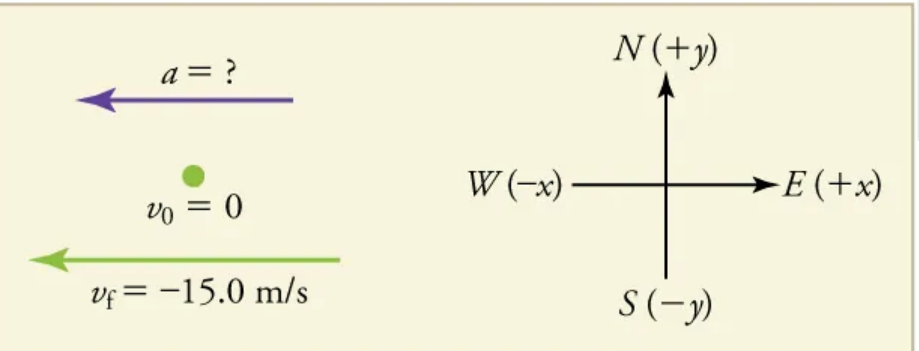



We can solve this problem by identifying $\Delta v$ and $\Delta t$ from the given information and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.

### Solution

1. Identify the knowns. $v_0 = 0$, $v_f = -15.0\ \text{m/s}$ (the negative sign indicates direction toward the west), $\Delta t = 1.80\ \text{s}$.

2. Find the change in velocity. Since the horse is going from zero to $-15.0\ \text{m/s}$, its change in velocity equals its final velocity: $\Delta v = v_f = -15.0\ \text{m/s}$.

3. Plug in the known values ($\Delta v$ and $\Delta t$) and solve for the unknown $\bar{a}$.

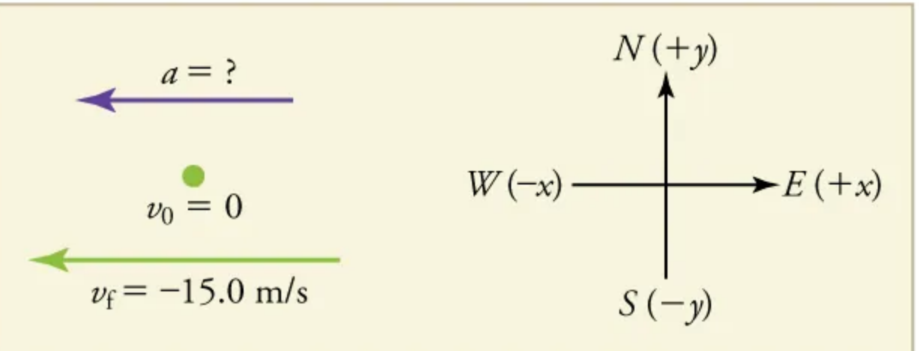
$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0\ \text{m/s}}{1.80\ \text{s}} = -8.33\ \text{m/s}^2. \tag{2.11}$$

## Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

### Discussion

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of $8.33\ \mathrm{m/s}^2$ due west means that the horse increases its velocity by 8.33 m/s due west each second, that is, 8.33 meters per second per second, which we write as $8.33\ \mathrm{m/s}^2$. This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.



## EXAMPLE 2.2

### Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of Figure 2.18?

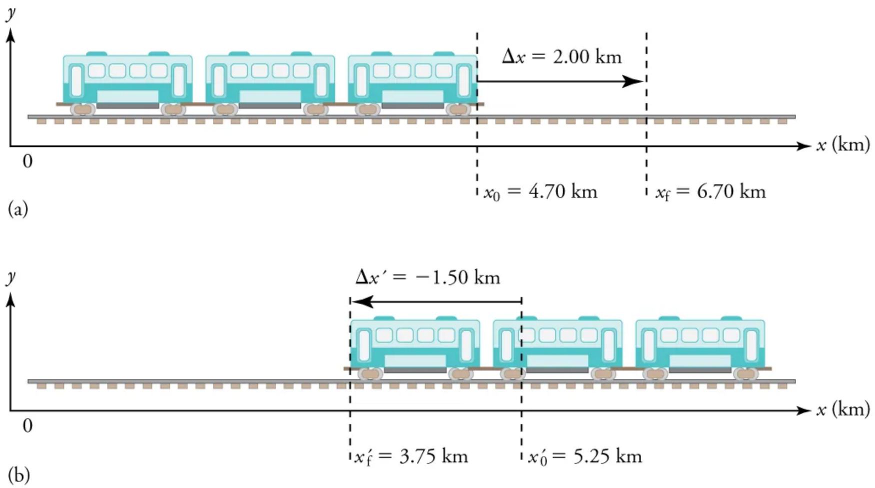



**Figure 2.18** One-dimensional motion of a subway train considered in Example 2.2, Example 2.3, Example 2.4, Example 2.5, Example 2.6, and Example 2.7. Here we have chosen the $x$-axis so that + means to the right and — means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from $x_0$ to $x_f$. Its displacement $\Delta x$ is +2.0 km. (b) The train moves to the left from $x'_0$ to $x'_f$. Its displacement $\Delta x'$ is $-1.5\ \text{km}$. (Note that the prime symbol (′) is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.)

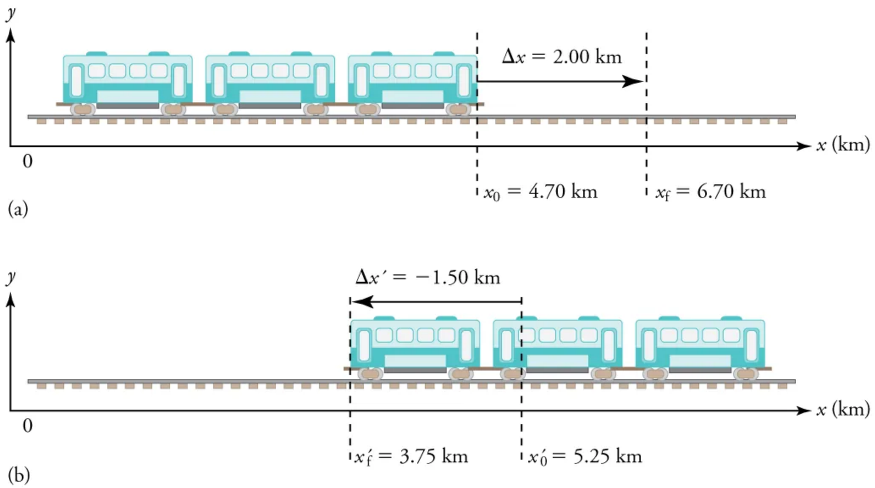
**EXAMPLE 2.2**

### Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of Figure 2.18?

#### Strategy

A drawing with a coordinate system is already provided, so we don't need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation $\Delta x = x_\mathrm{f} - x_0$. This is straightforward since the initial and final positions are given.

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## EXAMPLE 2.2

### Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of Figure 2.18?

**Solution**

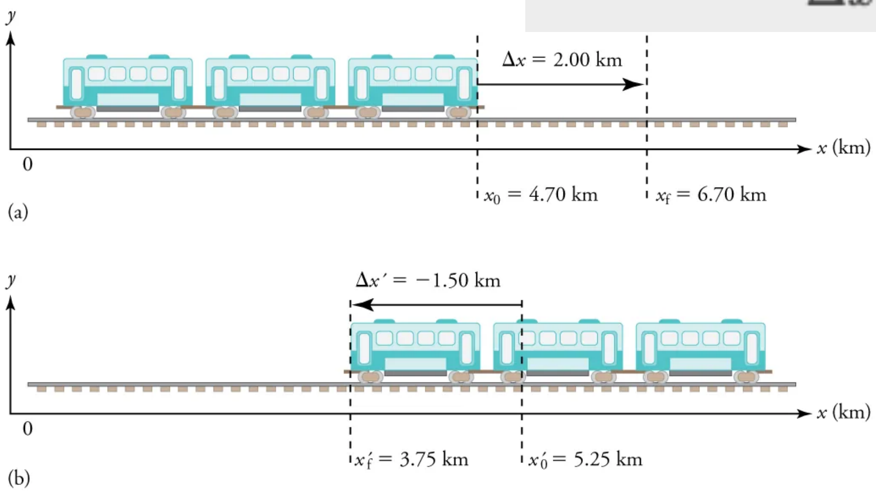
1. Identify the knowns. In the figure we see that $x_f = 6.70\text{ km}$ and $x_0 = 4.70\text{ km}$ for part (a), and $x'_f = 3.75\text{ km}$ and $x'_0 = 5.25\text{ km}$ for part (b).

2. Solve for displacement in part (a).

$$\Delta x = x_f - x_0 = 6.70\text{ km} - 4.70\text{ km} = +2.00\text{ km} \tag{2.12}$$

3. Solve for displacement in part (b).

$$\Delta x' = x'_f - x'_0 = 3.75\text{ km} - 5.25\text{ km} = -1.50\text{ km} \tag{2.13}$$

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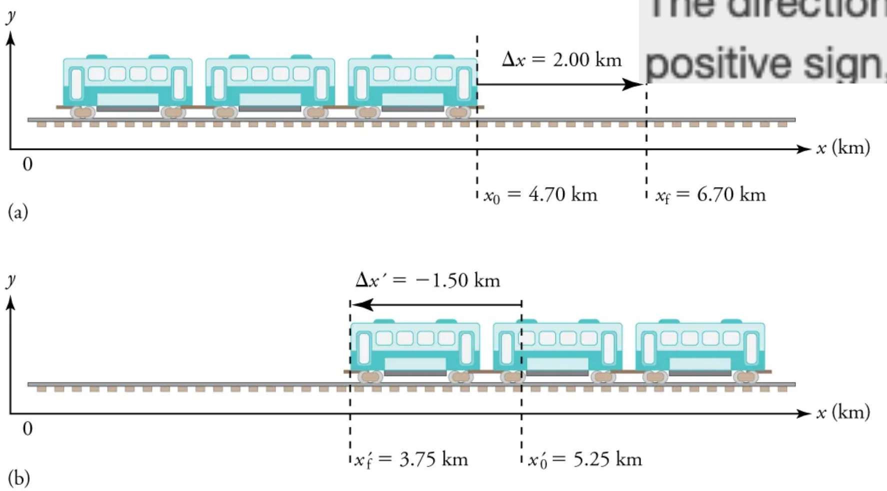
$$\Delta x' = x'_{\mathrm{f}} - x'_0 = 3.75 \text{ km} - 5.25 \text{ km} = -1.50 \text{ km} \tag{2.13}$$

### Discussion

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a negative sign.

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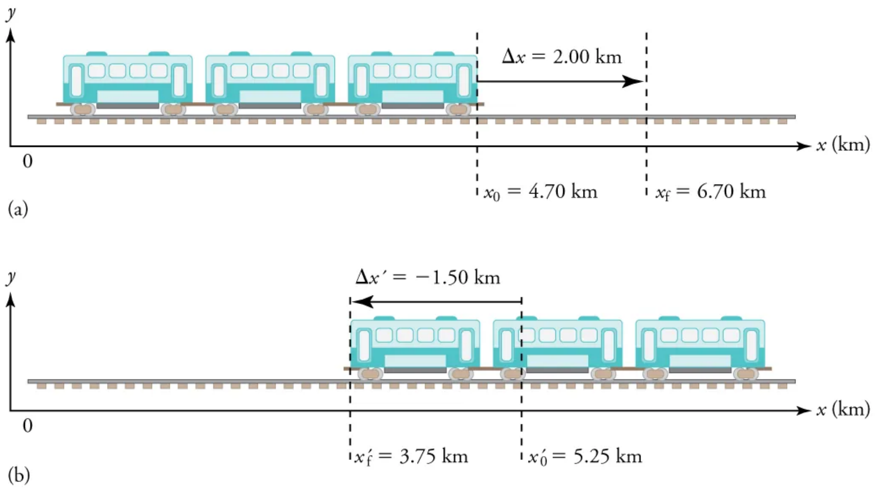
## EXAMPLE 2.3

### Comparing Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in Figure 2.18?

### Strategy

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in Example 2.2. Distance traveled is the total length of the path traveled between the two positions. (See Displacement.) In the case of the subway train shown in Figure 2.18, the distance traveled is the same as the distance between the initial and final positions of the train.

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## EXAMPLE 2.3

### Comparing Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in Figure 2.18?

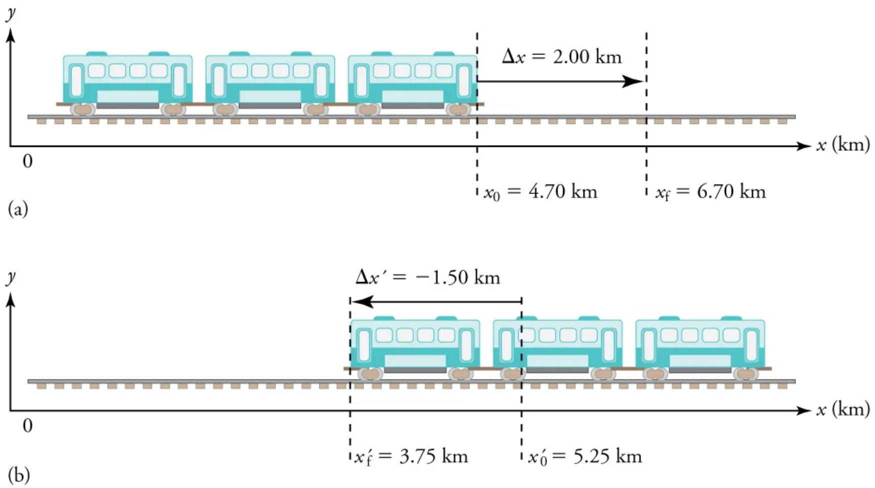
### Solution

1. The displacement for part (a) was +2.00 km. Therefore, the distance between the initial and final positions was 2.00 km, and the distance traveled was 2.00 km.

2. The displacement for part (b) was $-1.5\ \text{km}$. Therefore, the distance between the initial and final positions was 1.50 km, and the distance traveled was 1.50 km.

### Discussion

Distance is a scalar. It has magnitude but no sign to indicate direction.

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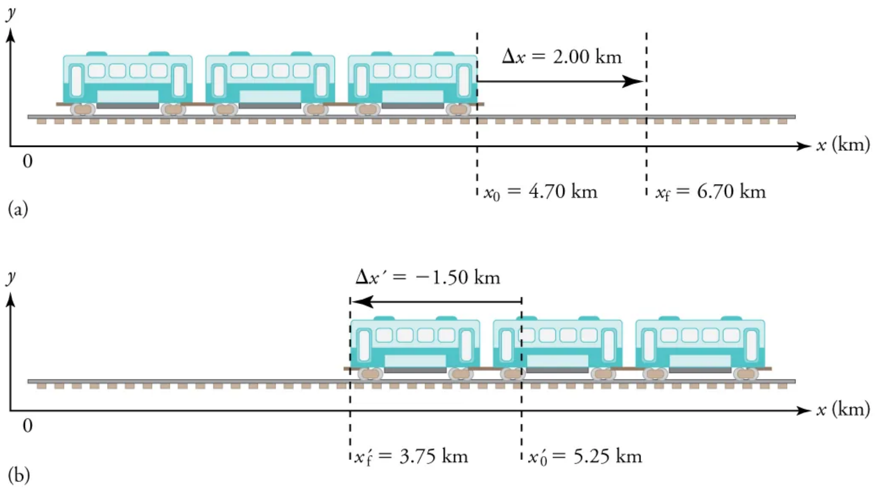


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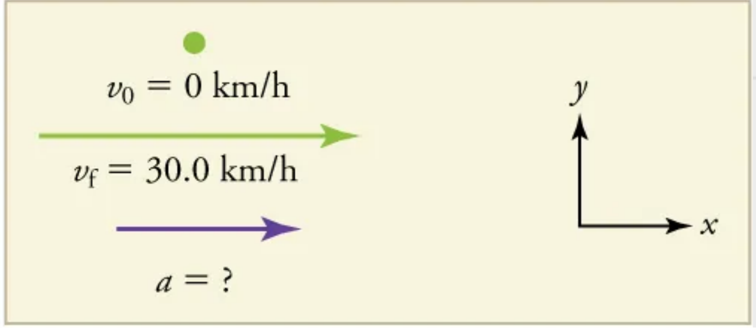
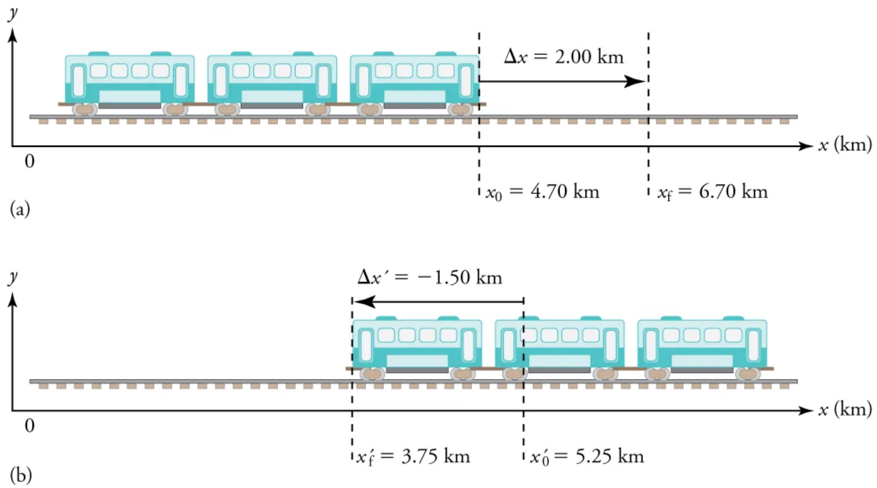



## EXAMPLE 2.4

### Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in Figure 2.18(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

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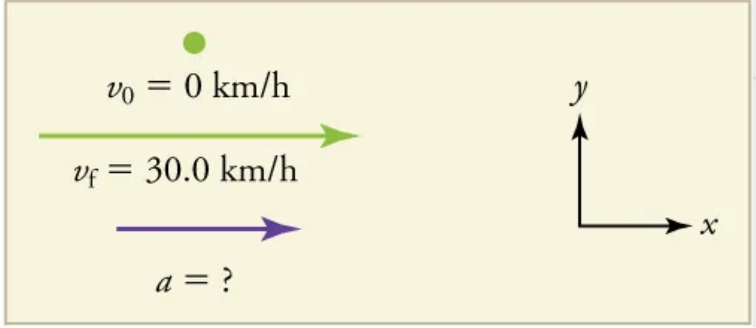


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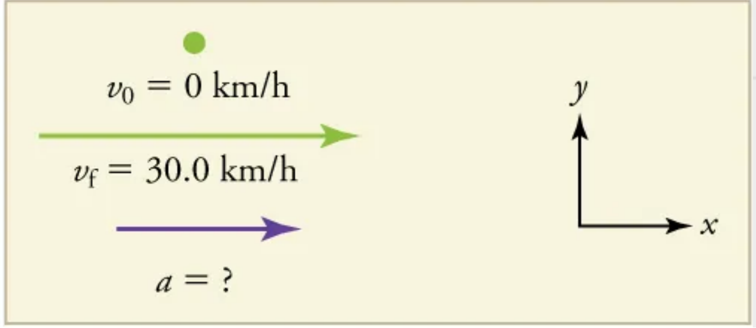
### Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in Figure 2.18(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

### Solution

1. Identify the knowns. $v_0 = 0$ (the trains starts at rest), $v_f = 30.0 \text{ km/h}$, and $\Delta t = 20.0 \text{ s}$.

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**Solution**

1. Identify the knowns. $v_0 = 0$ (the trains starts at rest), $v_f = 30.0\ \text{km/h}$, and $\Delta t = 20.0\ \text{s}$.

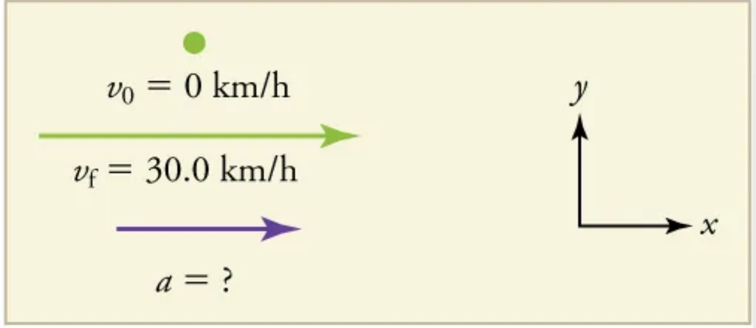
2. Calculate $\Delta v$. Since the train starts from rest, its change in velocity is $\Delta v = +30.0\ \text{km/h}$, where the plus sign means velocity to the right.

3. Plug in known values and solve for the unknown, $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+30.0\ \text{km/h}}{20.0\ \text{s}} \tag{2.14}$$

4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds. (See Physical Quantities and Units for more guidance.)

$$\bar{a} = \left(\frac{+30\ \text{km/h}}{20.0\ \text{s}}\right)\left(\frac{10^3\ \text{m}}{1\ \text{km}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = 0.417\ \text{m/s}^2 \tag{2.15}$$




**EXAMPLE 2.4**

### Solution

1. Identify the knowns. $v_0 = 0$ (the trains starts at rest), $v_f = 30.0\ \mathrm{km/h}$, and $\Delta t = 20.0\ \mathrm{s}$.

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### Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the *change* in velocity, as is always the case.

# Chapter 2

## 2.2 Vectors, Scalars, and Coordinate Systems

- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

# Chapter 2

## Vectors in Two Dimensions

A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction of a vector relative to some reference frame (i.e., coordinate system), using an arrow having length proportional to the vector's magnitude and pointing in the direction of the vector.

Figure 3.8 shows such a *graphical representation of a vector*, using as an example the total displacement for the person walking in a city considered in Kinematics in Two Dimensions: An Introduction. We shall use the notation that a boldface symbol, such as $\mathbf{D}$, stands for a vector. Its magnitude is represented by the symbol in italics, $D$, and its direction by $\theta$.

# Chapter 2

# Vectors

# Vectors

**SIDE NOTE**

The *magnitude* of a vector is similar to the *absolute value* of a real number. Like absolute value, the magnitude of a vector cannot be negative.

## Vectors

Many physical quantities such as length, area, volume, mass, and temperature are completely described by their magnitudes in appropriate units. Such quantities are called **scalar quantities**. Other physical quantities such as velocity, acceleration, and force are completely described only if *both* a magnitude (size) and a direction are specified. For example, the movement of wind is usually described by its speed (magnitude) and its direction, say, 15 miles per hour southwest. The wind speed and wind direction together form a **vector quantity** called the *wind velocity*.

**Objective 1 ▸**

## Geometric Vectors

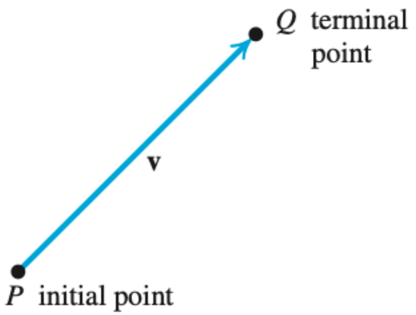



**Figure 7.21 ▸** A vector

We can represent a **vector** geometrically by a directed line segment with an arrowhead. The arrow specifies the direction of the vector, and its length describes the magnitude. The tail of the arrow is the vector's **initial point**, and the tip of the arrow is its **terminal point**. We denote vectors by lowercase boldface type, such as **a**, **b**, **i**, **j**, **u**, **v**, and **w**. When discussing vectors, we refer to real numbers as **scalars**. Scalars will be denoted by lowercase italic type, such as $a$, $b$, $x$, $y$, and $z$.

As in Figure 7.21, if the initial point of a vector **v** is $P$ and the terminal point is $Q$, we write the vector **v** as follows:

$$\mathbf{v} = \overrightarrow{PQ}.$$

The **magnitude** (or **norm**) of a vector $\mathbf{v} = \overrightarrow{PQ}$, denoted by $\|\mathbf{v}\|$ or $\left\|\overrightarrow{PQ}\right\|$, is the length of the vector **v** and is a scalar quantity.

**SIDE NOTE**

# Vectors

SIDE NOTE

For handwritten vector notation where bold print may not be possible, an arrow is placed over Lower case, italic letter.

the vector **v** and is a scalar quantity.

## Equivalent Vectors

Two vectors having the same length and same direction are called **equivalent vectors**. Because a vector is determined by its length and direction only, equivalent vectors are regarded as **equal** even though they may be located in different positions. If **v** and **w** are equivalent, we write $\mathbf{v} = \mathbf{w}$. See Figure 7.22.

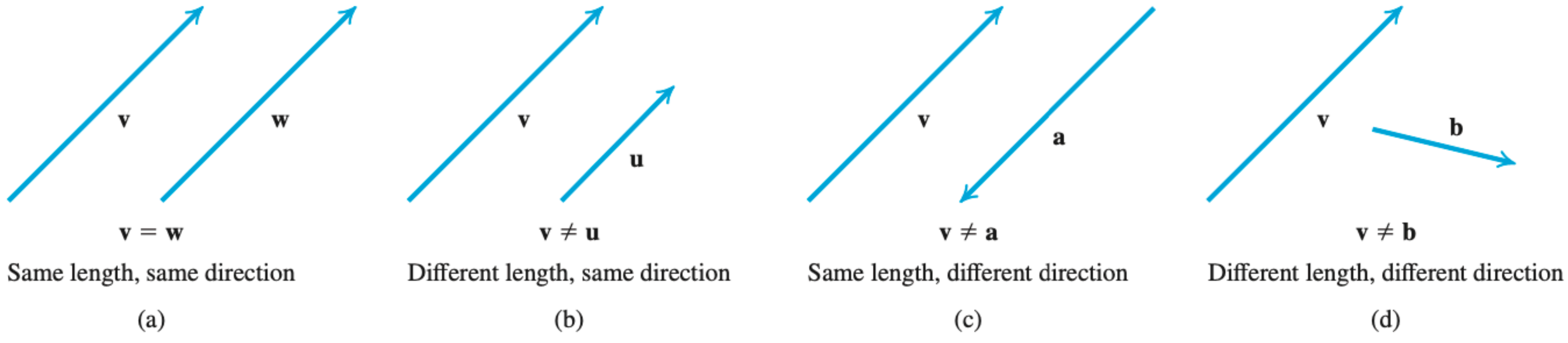



**Figure 7.22** ▸ Equivalent vectors have the same length and direction.

The vector of length zero is called the **zero vector** and is denoted by **0**. The zero vector has zero magnitude and arbitrary direction. If vectors **v** and **a**, as in Figure 7.22(c), have the same length and opposite direction, then **a** is the **opposite vector** of **v**, and we write $\mathbf{a} = -\mathbf{v}$.

# Vectors

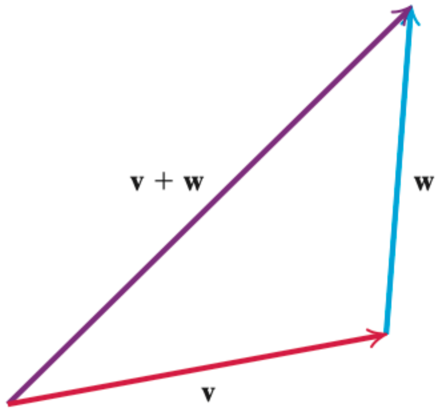



**Figure 7.23 ▸** Adding **v** and **w**

## Adding Vectors

We add two vectors **v** and **w** as shown in Figure 7.23.

### Geometric Vector Addition

Let **v** and **w** be any two vectors. Position the terminal point of **v** so that it coincides with the initial point of **w**. The *sum* **v** + **w** is the **resultant vector** whose initial point coincides with the initial point of **v**, and its terminal point coincides with the terminal point of **w**.

# Vectors

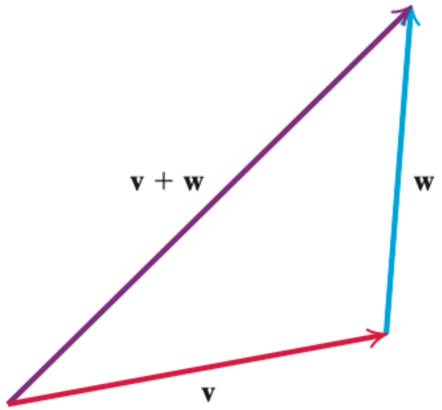



Figure 7.23 ▸ Adding **v** and **w**

## Adding Vectors

We add two vectors **v** and **w** as shown in Figure 7.23.

### Geometric Vector Addition

Let **v** and **w** be any two vectors. Position the terminal point of **v** so that it coincides with the initial point of **w**. The *sum* **v** + **w** is the **resultant vector** whose initial point coincides with the initial point of **v**, and its terminal point coincides with the terminal point of **w**.

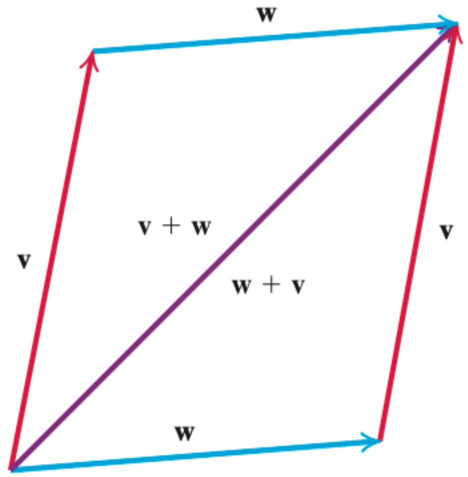



Figure 7.24 ▸ $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

In Figure 7.24, we have constructed two sums, $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$. It shows that

$$\boxed{\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}}$$

and that the sum coincides with the diagonal of the parallelogram determined by **v** and **w** when **v** and **w** have the same initial point.

Vector subtraction is defined just like the subtraction of real numbers. For any two vectors **v** and **w**, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$, where $-\mathbf{w}$ is the opposite of **w**. See Figure 7.25.

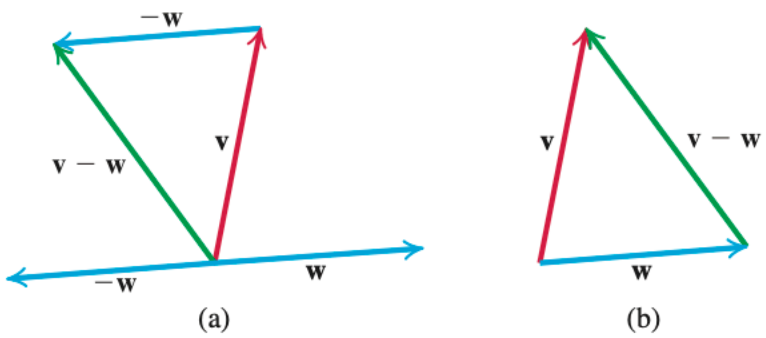



Figure 7.25 ▸ Vector $\mathbf{v} - \mathbf{w}$

Figure 7.25(b) shows that you can construct the difference vector $\mathbf{v} - \mathbf{w}$ without first drawing $-\mathbf{w}$: place the vectors **v** and **w** so that their initial points coincide. Then the vector from the terminal point of **w** to the terminal point of **v** is the vector $\mathbf{v} - \mathbf{w}$.

# Vectors

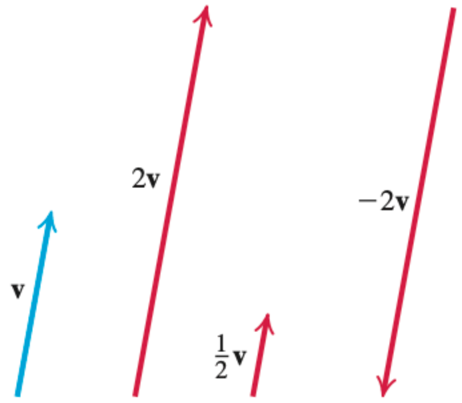



**Figure 7.26** ▸ Scalar multiples of **v**

drawing $-\mathbf{w}$: place the vectors $\mathbf{v}$ and $\mathbf{w}$ so that their initial points coincide. Then the vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$ is the vector $\mathbf{v} - \mathbf{w}$.

A second basic arithmetic operation for vectors is multiplying vectors by real numbers.

### Scalar Multiples of Vectors

Let $\mathbf{v}$ be a vector and $c$ a scalar (a real number). The vector $c\mathbf{v}$ is called the **scalar multiple** of $\mathbf{v}$. See Figure 7.26.

If $c > 0$, $c\mathbf{v}$ has the same direction as $\mathbf{v}$ and magnitude $c\|\mathbf{v}\|$.
If $c < 0$, $c\mathbf{v}$ has the opposite direction of $\mathbf{v}$ and magnitude $|c|\,\|\mathbf{v}\|$.
If $c = 0$, $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.

# Vectors

# Vectors

# Vectors

# Vectors

# Physics systems, solutions

# Physics systems, solutions

In order to analyze and determine solutions of Physics systems, we need to understand:

- Linear equations, also known as linear functions.
- The slope of the lines, the standard form.
- The negative reciprocal slopes.
- Parallel lines.
- Perpendicular lines.

P3 sec

**Linear Functions** A function of the form $f(x) = mx + b$, where $m$ and $b$ are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

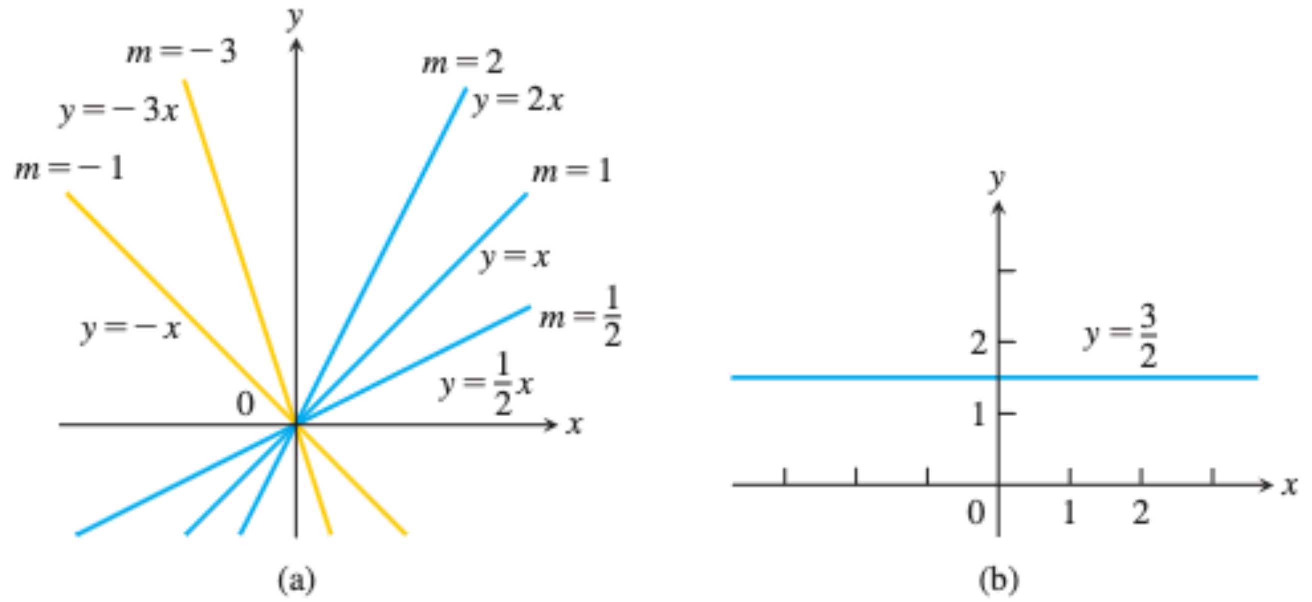



**FIGURE 1.14** (a) Lines through the origin with slope $m$. (b) A constant function with slope $m = 0$.

> **DEFINITION** Two variables $y$ and $x$ are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant $k$.

# P3 section review.

## Special-Product Formulas

$A$ and $B$ represent any algebraic expressions.

| Formula | Example |
|---|---|
| *Product giving a difference of squares* | |
| $(A+B)(A-B)=A^2-B^2$ | $(5x+2)(5x-2)=(5x)^2-2^2$<br>$=25x^2-4$ |
| *Squaring a binomial sum or difference* | |
| $(A+B)^2=A^2+2AB+B^2$ | $(3x+2)^2=(3x)^2+2(3x)(2)+2^2$<br>$=9x^2+12x+4$ |
| $(A-B)^2=A^2-2AB+B^2$ | $(2x-5)^2=(2x)^2-2(2x)(5)+5^2$<br>$=4x^2-20x+25$ |
| *Cubing a binomial sum or difference* | |
| $(A+B)^3=A^3+3A^2B+3AB^2+B^3$ | $(x+5)^3=x^3+3x^2(5)+3x(5)^2+5^3$<br>$=x^3+15x^2+75x+125$ |
| $(A-B)^3=A^3-3A^2B+3AB^2-B^3$ | $(x-4)^3=x^3-3x^2(4)+3x(4)^2-4^3$<br>$=x^3-12x^2+48x-64$ |
| *Products giving a sum or difference of cubes* | |
| $(A+B)(A^2-AB+B^2)=A^3+B^3$ | $(x+2)(x^2-2x+4)=x^3+2^3=x^3+8$ |
| $(A-B)(A^2+AB+B^2)=A^3-B^3$ | $(x-3)(x^2+3x+9)=x^3-3^3=x^3-27$ |

# P3 section review.

**Special-Product Formulas**

$A$ and $B$ represent any algebraic expressions.

| Formula | Example |
| --- | --- |
| *Product giving a difference of squares* | |
| $(A+B)(A-B) = A^2 - B^2$ | $(5x+2)(5x-2) = (5x)^2 - 2^2$<br>$= 25x^2 - 4$ |
| *Squaring a binomial sum or difference* | |
| $(A+B)^2 = A^2 + 2AB + B^2$ | $(3x+2)^2 = (3x)^2 + 2(3x)(2) + 2^2$<br>$= 9x^2 + 12x + 4$ |
| $(A-B)^2 = A^2 - 2AB + B^2$ | $(2x-5)^2 = (2x)^2 - 2(2x)(5) + 5^2$<br>$= 4x^2 - 20x + 25$ |
| *Cubing a binomial sum or difference* | |
| $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ | $(x+5)^3 = x^3 + 3x^2(5) + 3x(5)^2 + 5^3$<br>$= x^3 + 15x^2 + 75x + 125$ |
| $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ | $(x-4)^3 = x^3 - 3x^2(4) + 3x(4)^2 - 4^3$<br>$= x^3 - 12x^2 + 48x - 64$ |
| *Products giving a sum or difference of cubes* | |
| $(A+B)(A^2 - AB + B^2) = A^3 + B^3$ | $(x+2)(x^2 - 2x + 4) = x^3 + 2^3 = x^3 + 8$ |
| $(A-B)(A^2 + AB + B^2) = A^3 - B^3$ | $(x-3)(x^2 + 3x + 9) = x^3 - 3^3 = x^3 - 27$ |

# 1.1 section review.

## Solving Linear Equations in One Variable

### Linear Equations

**A conditional linear equation in one variable**, such as $x$, is an equation that can be written in the **standard form**

$$ax + b = 0,$$

where $a$ and $b$ are real numbers with $a \neq 0$.

### SIDE NOTE

When finding the domain of a variable, remember that

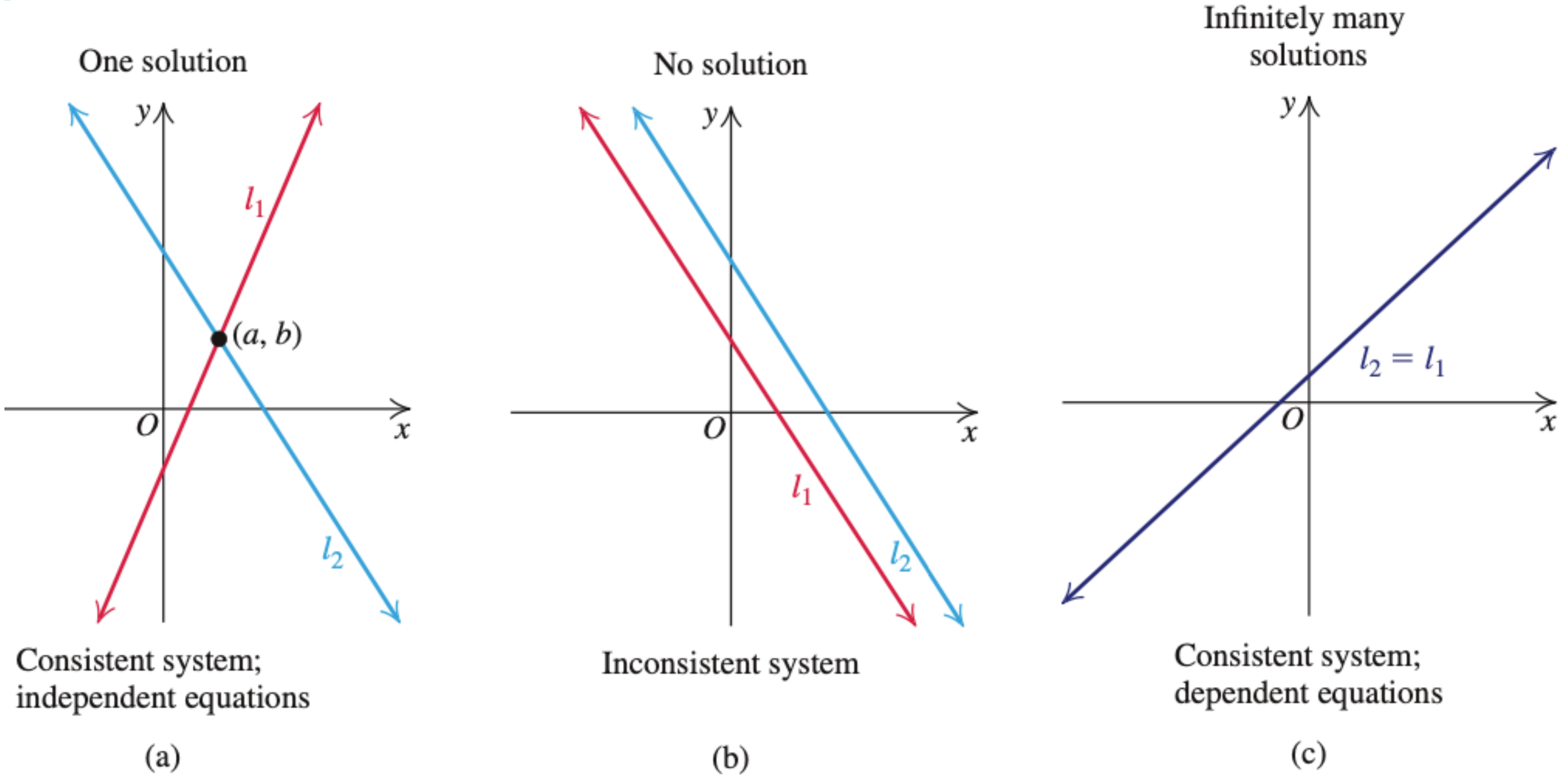
1. Division by 0 is undefined.
2. The square root (or any even root) of a negative number is not a real number.

### Identifying Types of Linear Equations

1. An equation that is satisfied by all values in the domain of the variable is an **identity**. For example, $2(x - 1) = 2x - 2$. When you try to solve an identity, you get a true statement that does not depend on the variable, such as $3 = 3$ or $0 = 0$.
2. An equation that is not an identity but is satisfied by at least one number in the domain of the variable is a **conditional** equation. For example, $2x = 6$ is a linear conditional equation; $x = 3$ is a solution, but $x = 0$ is not.
3. An equation that is not satisfied by any value of the variable is an **inconsistent** equation. For example, $x = x + 2$ is an inconsistent equation. Because no number is 2 more than itself, the solution set of the equation $x = x + 2$ is $\varnothing$. When you try to solve an inconsistent equation, you obtain a false statement such as $0 = 2$.

# 1.1 section

1. One solution, also called **unique solution** (the lines intersect; see Figure 8.2(a)): the system is consistent, and the equations in the system are said to be **independent**.
2. No solution (the lines are parallel; see Figure 8.2(b)): The system is inconsistent.
3. Infinitely many solutions (the lines coincide; see Figure 8.2(c)): The system is consistent, and the equations in the system are said to be **dependent**.




$l_1 \rightarrow y = x - 1, l_2 \rightarrow y = -x + 2,$ $l_1 \rightarrow y = -x, l_2 \rightarrow y = -x + 2,$ $l_1 = l_2 := y = x\,.$

# 1.1 sectio

**PROCEDURE IN ACTION**

## EXAMPLE 3 The Substitution Method

| Objective | Example |
|---|---|
| *Reduce the solution of the system to the solution of one equation in one variable by substitution.* | *Solve the system:* $\begin{cases} 2x - 5y = 3 & (1) \\ y - 2x = 9 & (2). \end{cases}$ |
| **Step 1 ▸ Solve for one variable.** Choose one of the equations and express one of its variables in terms of the other variable. | **1.** We choose equation (2) and express $y$ in terms of $x$.<br>$y = 2x + 9 \quad (3)$ — Solve equation (2) for $y$. |
| **Step 2 ▸ Substitute.** Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable. | **2.** $2x - 5y = 3$ — Equation (1)<br>$2x - 5(2x + 9) = 3$ — Substitute $2x + 9$ for $y$.<br>$2x - 10x - 45 = 3$ — Distributive property |
| **Step 3 ▸ Solve** the equation obtained in Step 2. | **3.** $-8x - 45 = 3$ — Combine like terms.<br>$-8x = 3 + 45$ — Add 45 to both sides.<br>$x = -6$ — Solve for $x$. |
| **Step 4 ▸ Back-substitute.** Substitute the value(s) you found in Step 3 back into the expression you found in Step 1. The result is the solution(s). | **4.** $y = 2x + 9$ — Equation (3) from Step 1<br>$y = 2(-6) + 9 = -3$ — Substitute $x = -6$.<br>Because $x = -6$ and $y = -3$, the solution set is $\{(-6, -3)\}$. |
| **Step 5 ▸ Check.** Check your answer(s) in the original equations. | **5.** Check: $x = -6$ and $y = -3$.<br>Equation (1): $2(-6) - 5(-3) \stackrel{?}{=} 3$; $-12 + 15 = 3$ ✓<br>Equation (2): $-3 - 2(-6) \stackrel{?}{=} 9$; $-3 + 12 = 9$ ✓ |

**Practice Problem 3.** Solve: $\begin{cases} x - y = 5 & (1) \\ 2x + y = 7 & (2). \end{cases}$

# 1.1 section review.

**104.** **Investment.** Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

# 1.1 section.

**104. Investment.** Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1) + y (investment 2) = ?

# 1.1 section.

**104.** **Investment.** Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.

# 8.1 section.

**104. Investment.** Mr. Sharma invested a total of \$30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of \$2550 from both investments. How much did he invest at each rate?

- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.
- $.08x + .105y = \$2550$.
- Then: $8x + 10.5y = \$255000$.

# 8.1 section.

**104. Investment.** Mr. Sharma invested a total of $30,000 in two ventures for a year. The annual return from one of them was 8%, and the other paid 10.5% for the year. He received a total income of $2550 from both investments. How much did he invest at each rate?

- x (investment 1 in \$) + y (investment 2 in \$) = \$30000.
- .08x + .105y = \$2550.
- Then: 8x + 10.5y = \$255000.
- Substitute: 8( \$30000 – y) + 10.5y = \$255000.
- \$240000 – 8y + 10.5y = \$255000.
- 2.5y = \$15000.
- y = \$6000.
- Substitute into first, total equation:
- x + \$6000 = \$30000.
- x = \$24000.

# Motivation.

What is mathematics?

What is math?

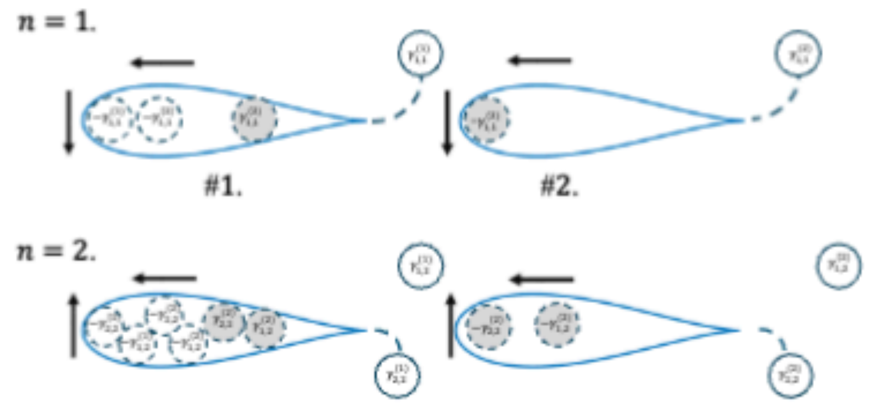



**Figure 2.3** Schematic of the flow-induced model configuration for $M = 2$ wings and their shed vortices in two time steps ($n = 1$ and $n = 2$). The velocity potential around wing 1 is considered, so the second wing is only represented by its bound vorticity (gray circles). The dotted circles represent image vortices. The dashed curves connect the trailing edges of each wing to the vortex most recently shed. The bound vorticity before the vortex is shed is $\Gamma_{1,1}^{(1)} = 0$, $\Gamma_{1,1}^{(2)} = 0$.

step is

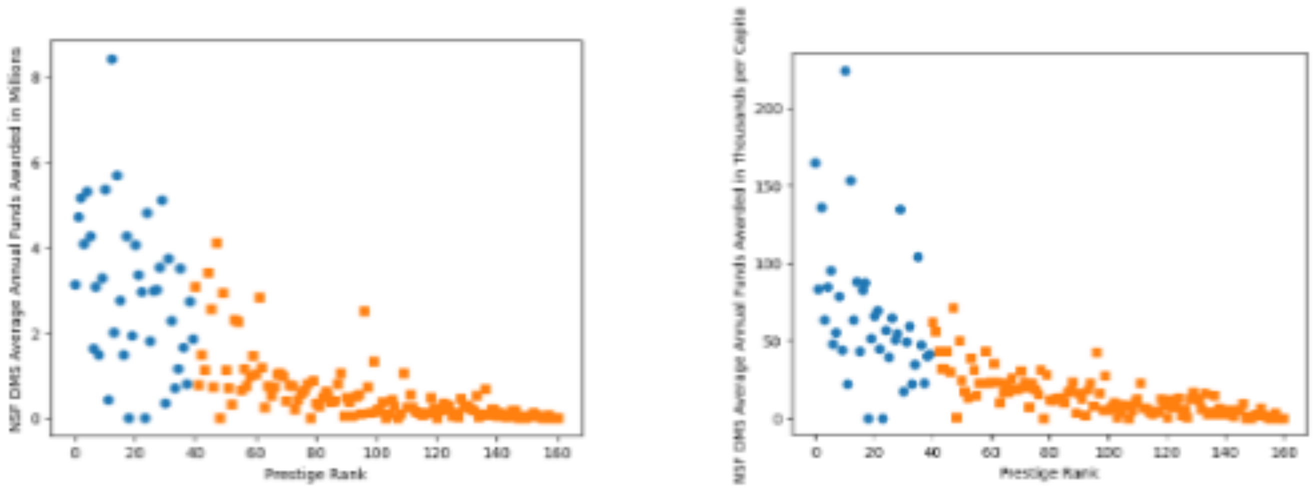
$$\begin{aligned}
w_n^{(j)}(\zeta) &= \overline{W}_n^{(j)}(J(\zeta) - \zeta) - \frac{r^2 W_n^{(j)}}{\zeta} - i\frac{1}{2\pi}\sum_{l=1}^{M}\sum_{k=1}^{n}\gamma_{k,n}^{(l)} \log\left(\frac{-r}{\zeta_{k,n}^{(j,l)}}\frac{\zeta - \zeta_{k,n}^{(j,l)}}{\zeta - r^2/\overline{\zeta}_{k,n}^{(j,l)}}\right) \\
&- i\frac{1}{2\pi}\sum_{l\neq j}^{M}\left(\Gamma_n^{(l)} - \gamma_n^{(l)}\right)\log\left(\frac{-r}{\chi_n^{(j,l)}}\frac{\zeta - \chi_n^{(j,l)}}{\zeta - r^2/\overline{\chi}_n^{(j,l)}}\right) \\
&- i\frac{1}{2\pi}\left[\sum_{l=1}^{M}\left(\Gamma_n^{(l)} + \sum_{k=1}^{n-1}\gamma_{k,n}^{(l)}\right)\right]\log\left(\frac{\zeta}{r}\right), \\
\zeta_{k,n}^{(j,l)} &= J^{-1}\left(z_{k,n}^{(l)} - Z_n^{(j)}\right), \quad \chi_n^{(j,l)} = J^{-1}\left(Z_n^{(l)} - Z_n^{(j)}\right). \qquad (2.23)
\end{aligned}$$

That is, $\zeta_{k,n}^{(j,l)}$ is the position of vortex $z_{k,n}^{(l)}$ at time $n$ in the $\zeta$-plane, which was shed by wing $l$ at time step $k$, using coordinates where wing $j$ is at the center. Similarly, $\chi_n^{(j,l)}$ is the position of wing $l$ at time $n$, again using coordinates where wing $j$ is at the

# Motivation.

## What is mathematics?

Quantifying & Documenting Inequalities in Math 11




(a) Total funding

(b) Funding per faculty

Fig. 4: Average annual grant funding for Mathematics departments by the prestige rank of the department, displayed by total funding to the department (a), and on a per capita basis accounting for the varying number of faculty in each department (b).

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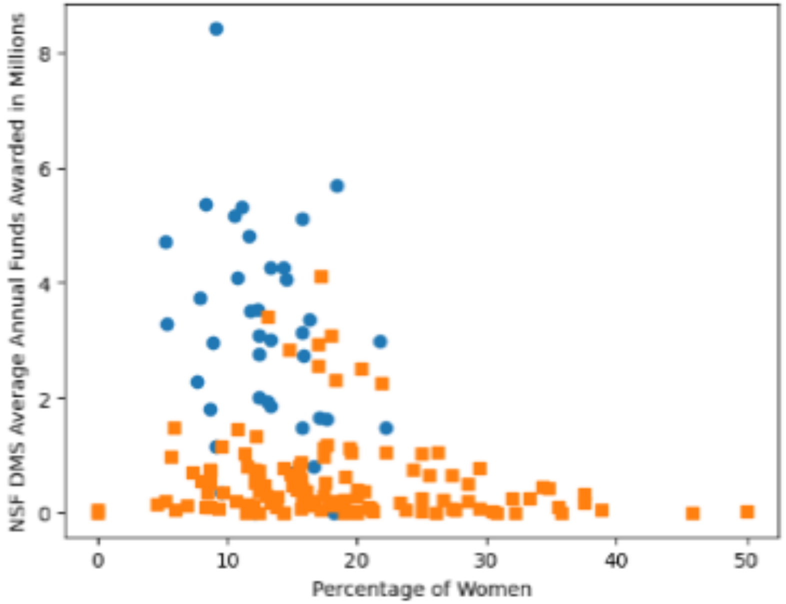



Fig. 5: Average annual grant funding for Mathematics departments by the percentage of women faculty in the department. The upper quartile (by prestige) departments are represented by blue circles, while the remainder are orange squares.

# Motivation.

What does math have to do with engineering?

Can we think of different kinds of engineers?

How does math scaffold in the daily duties of an engineer ?

# Presentation and communication.

Determine the values of ‘x’ that satisfy this equation:

$$216x^3 - 8 = 0,$$

# Presentation and communication.

Determine and describe the reasons the Roman Empire dominated their regions from 27 BC to 476 AC?

# Presentation and communication.

Determine and describe the reasons the Roman Empire dominated their regions from 27 BC to 476 AC?

The Roman Empire dominated its regions through a combination of overwhelming military superiority, highly organized administrative infrastructure, and strategic cultural assimilation. Professionally trained legions and advanced engineering allowed Rome to conquer and secure vast territories, while an extensive network of roads and centralized governance facilitated efficient trade and communication. Furthermore, by extending citizenship and incorporating local elites into their administrative framework, the Romans successfully fostered regional stability and long-term loyalty across diverse populations.

# Presentation and communication.

Determine the values of ‘x’ that satisfy this equation:

$$216x^3 - 8 = 0,$$

# Presentation and communication.

Determine the values of 'x' that satisfy this equation:

$$216x^3 - 8 = 0,$$

$$8\left(27x^3 - 1\right) = 0,$$

$$\left(27x^3 - 1\right) = 0,$$

$$(3x - 1)\left(3x^2 + 6x + 1\right) = 0,$$

$$\implies (3x - 1) = 0, \left(3x^2 + 6x + 1\right) = 0,$$

$$\implies x = \{\frac{1}{3}, \frac{\sqrt{6}}{3} - 1, -\frac{\sqrt{6}}{3} - 1\}$$

# Presentation and communication.

Determine the values of ‘x’ that satisfy this equation:

$$216x^3 - 8 = 0,$$

$$8\left(27x^3 - 1\right) = 0, \ \textit{Distributive Prop.}$$

$$\left(27x^3 - 1\right) = 0, \ \textit{Mult. eq by } \frac{1}{8}$$

$$(3x - 1)\left(3x^2 + 6x + 1\right) = 0, \ \textit{Polynom. Factor.}$$

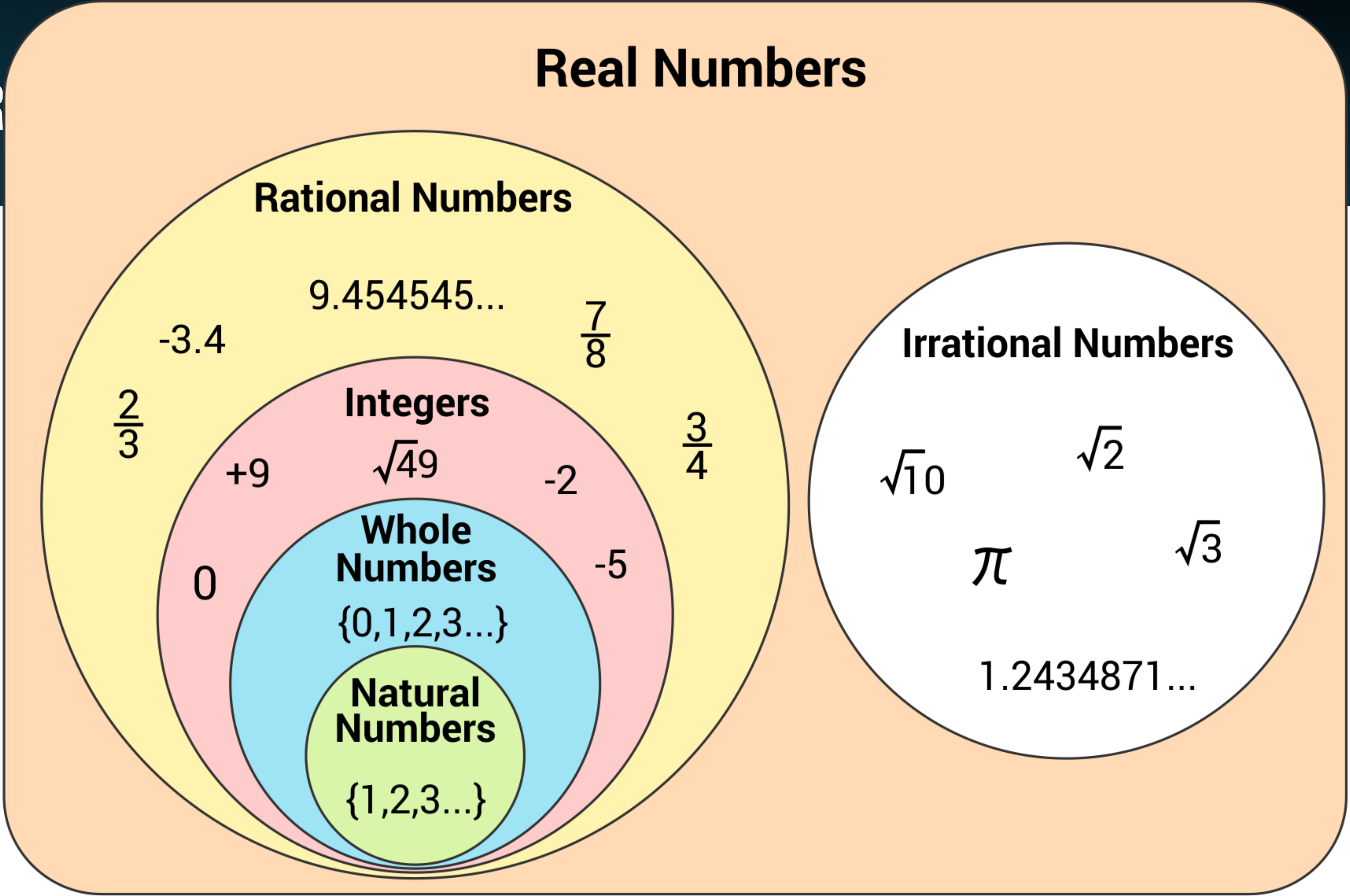
$$\implies (3x - 1) = 0, \left(3x^2 + 6x + 1\right) = 0, \textit{Zero Prop.}$$

$$\implies x = \{\frac{1}{3}, \frac{\sqrt{6}}{3} - 1, -\frac{\sqrt{6}}{3} - 1\}$$

# Review.

# Review.

R



Real Numbers
Rational Numbers
9.454545...
-3.4
$\frac{7}{8}$
$\frac{2}{3}$
$\frac{3}{4}$
Integers
+9
$\sqrt{49}$
-2
-5
0
Whole Numbers
{0,1,2,3...}
Natural Numbers
{1,2,3...}
Irrational Numbers
$\sqrt{10}$
$\sqrt{2}$
$\pi$
$\sqrt{3}$
1.2434871...

# Section 1.1

**Properties of the Real Numbers**

| Name | Addition (A) | Multiplication (M) | Examples |
|---|---|---|---|
| **Closure** | $a + b$ is a real number | $a \cdot b$ is a real number | A: $1 + \sqrt{2}$ is a real number.<br>M: $2 \cdot \pi$ is a real number. |
| **Commutative** | $a + b = b + a$ | $a \cdot b = b \cdot a$ | A: $4 + 7 = 7 + 4$<br>M: $3 \cdot 8 = 8 \cdot 3$ |
| **Associative** | $(a + b) + c = a + (b + c)$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ | A: $(2 + 1) + 7 = 2 + (1 + 7)$<br>M: $(5 \cdot 9) \cdot 13 = 5 \cdot (9 \cdot 13)$ |
| **Identity** | There is a unique real number 0, called the **additive identity**, such that $a + 0 = a$ and $0 + a = a$. | There is a unique real number 1, called the **multiplicative identity**, such that $a \cdot 1 = a$ and $1 \cdot a = a$. | A: $3 + 0 = 3$ and $0 + 3 = 3$<br>M: $7 \cdot 1 = 7$ and $1 \cdot 7 = 7$ |
| **Inverse** | There is a unique real number $-a$, called the **opposite of** $a$, such that $a + (-a) = 0$ and $(-a) + a = 0$. | If $a \neq 0$, there is a unique real number $\frac{1}{a}$, called the **reciprocal of** $a$, such that $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$. | A: $5 + (-5) = 0$ and $(-5) + 5 = 0$<br>M: $4 \cdot \frac{1}{4} = 1$ and $\frac{1}{4} \cdot 4 = 1$ |
| **Distributive** | Multiplication distributes over addition. | $a \cdot (b + c) = a \cdot b + a \cdot c$<br>$(a + b) \cdot c = a \cdot c + b \cdot c$ | $3 \cdot (2 + 5) = 3 \cdot 2 + 3 \cdot 5$<br>$(2 + 5) \cdot 3 = 2 \cdot 3 + 5 \cdot 3$ |

Section 1.1
Review.

Algebra –
fundamental.

# Algebra

## Properties of Real Numbers

| | |
|---|---|
| Commutative: | $a + b = b + a; \quad ab = ba$ |
| Associative: | $a + (b + c) = (a + b) + c;$ <br> $a(bc) = (ab)c$ |
| Additive Identity: | $a + 0 = 0 + a = a$ |
| Additive Inverse: | $-a + a = a + (-a) = 0$ |
| Multiplicative Identity: | $a \cdot 1 = 1 \cdot a = a$ |
| Multiplicative Inverse: | $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1, a \neq 0$ |
| Distributive: | $a(b + c) = ab + ac$ |

## Exponents and Radicals

$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn} \qquad (ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad a^{-n} = \frac{1}{a^n}$$

If $n$ is even, $\sqrt[n]{a^n} = |a|$.

If $n$ is odd, $\sqrt[n]{a^n} = a$.

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \quad a, b \geq 0$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$$

## Factoring Formulas

$a^2 - b^2 = (a + b)(a - b)$

$a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

## Interval Notation

$(a, b) = \{x | a < x < b\}$

$[a, b] = \{x | a \leq x \leq b\}$

$(a, b] = \{x | a < x \leq b\}$

$[a, b) = \{x | a \leq x < b\}$

$(-\infty, a) = \{x | x < a\}$

$(a, \infty) = \{x | x > a\}$

$(-\infty, a] = \{x | x \leq a\}$

$[a, \infty) = \{x | x \geq a\}$

## Absolute Value

$|a| \geq 0$

For $a > 0$,

$|X| = a \rightarrow X = -a \quad or \quad X = a,$

$|X| < a \rightarrow -a < X < a,$

$|X| > a \rightarrow X < -a \quad or \quad X > a.$

# Section 1.1

**SIDE NOTE**

You can remember the order of operations by the acronym **PEMDAS** (Please Excuse My Dear Aunt Sally):

- **P**arentheses
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction.

**The Order of Operations**

**Step 1 ▸** Whenever a **fraction bar** is encountered, work separately above and below it.

**Step 2 ▸** When **parentheses or other grouping symbols** are present, start inside the innermost pair of grouping symbols and work outward.

**Step 3 ▸** Do operations involving **exponents**.

**Step 4 ▸** Do **multiplications** and **divisions** in the order in which they occur, **working from left to right**.

**Step 5 ▸** Do **additions** and **subtractions** in the order in which they occur, **working from left to right**.

# Review.

# P6 section review.

**Square Root Property**

For $a \geq 0$, if $x^2 = a$ then $x = \pm\sqrt{a}$.

**The Exponent $\frac{1}{n}$**

For any real number $a$ and any integer $n > 1$,

$$a^{1/n} = \sqrt[n]{a}.$$

When $n$ is even and $a < 0$, $\sqrt[n]{a}$ and $a^{1/n}$ are not real numbers.
Notice that the denominator $n$ of the rational exponent is the index of the radical.

# P6 section review.

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For $a \geq 0$, if $x^2 = a$ then $x = \pm\sqrt{a}$.

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Notice that the denominator $n$ of the rational exponent is the index of the radical.

## Exponents and Radicals

$$a^m \cdot a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn} \qquad (ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad a^{-n} = \frac{1}{a^n}$$

If $n$ is even, $\sqrt[n]{a^n} = |a|$.

If $n$ is odd, $\sqrt[n]{a^n} = a$.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \quad a, b \geq 0$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$$

# P6 section review.

## Square Root Property

For $a \geq 0$, if $x^2 = a$ then $x = \pm\sqrt{a}$.

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For any real number $a$ and any integer $n > 1$,

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Notice that the denominator $n$ of the rational exponent is the index of the radical.

## Exponents and Radicals

$a^m \cdot a^n = a^{m+n}$ $\qquad$ $\dfrac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$ $\qquad$ $(ab)^m = a^m b^m$

$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ $\qquad$ $a^{-n} = \dfrac{1}{a^n}$

If $n$ is even, $\sqrt[n]{a^n} = |a|$.

If $n$ is odd, $\sqrt[n]{a^n} = a$.

$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}, \quad a, b \geq 0$

$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

$\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{m/n}$

**WARNING**

In general:

$\sqrt[n]{a^n + b^n} \neq a + b$

$\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

$\sqrt[n]{a} + \sqrt[n]{b}$ cannot be added if $a \neq b$.

$\sqrt[n]{a} + \sqrt[n]{a} = 2\sqrt[n]{a}$

$\sqrt[n]{a} + \sqrt[m]{a}$ cannot be added if $n \neq m$.

# P6 section review.

### The Exponent $\frac{1}{n}$ and Radicals

If $a$ is any real number and $n$ is any integer greater than 1, then

| | **$n$ even** | **$n$ odd** |
|---|---|---|
| If $a > 0$, | $\sqrt[n]{a} = a^{1/n}$ is positive. | $\sqrt[n]{a} = a^{1/n}$ is positive. |
| If $a < 0$, | $\sqrt[n]{a} = a^{1/n}$ is not a real number. | $\sqrt[n]{a} = a^{1/n}$ is negative. |
| If $a = 0$, | $\sqrt[n]{0} = 0^{1/n} = 0.$ | $\sqrt[n]{0} = 0^{1/n} = 0.$ |

### Rational Exponents

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m},$$

provided that $m$ and $n$ are integers with no common factors, $n > 1$, and $\sqrt[n]{a}$ is a real number.

# P6 section review.

## Properties of Rational Exponents

If $r$ and $s$ are rational numbers and $a$ and $b$ are real numbers, then

$$a^r \cdot a^s = a^{r+s} \qquad (a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r$$

$$\frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad a^{-r} = \frac{1}{a^r} \qquad \left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^r$$

provided that all of the expressions used are defined.

## Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say $y$, depends on the value of another variable quantity, which we often call $x$. We say that "$y$ is a function of $x$" and write this symbolically as

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol $f$ represents the function, the letter $x$ is the **independent variable** representing the input value to $f$, and $y$ is the **dependent variable** or output value of $f$ at $x$.

> **DEFINITION** A **function** $f$ from a set $D$ to a set $Y$ is a rule that assigns a *unique* value $f(x)$ in $Y$ to each $x$ in $D$.

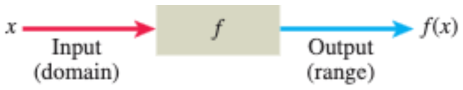


**FIGURE 1.1** A diagram showing a function as a kind of machine.

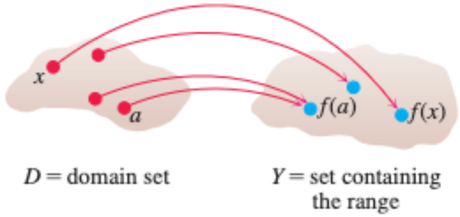


**FIGURE 1.2** A function from a set $D$ to a set $Y$ assigns a unique element of $Y$ to each element in $D$.

**EXAMPLE 1** Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of $x$ for which the formula makes sense.

| Function | Domain ($x$) | Range ($y$) |
|---|---|---|
| $y = x^2$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $y = 1/x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $y = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $y = \sqrt{4 - x}$ | $(-\infty, 4]$ | $[0, \infty)$ |
| $y = \sqrt{1 - x^2}$ | $[-1, 1]$ | $[0, 1]$ |

## Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of $y$ with respect to $x$ over the interval $[x_1, x_2]$ by dividing the change in the value of $y$, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol $h$ for $\Delta x$ to simplify the notation here and later on.)

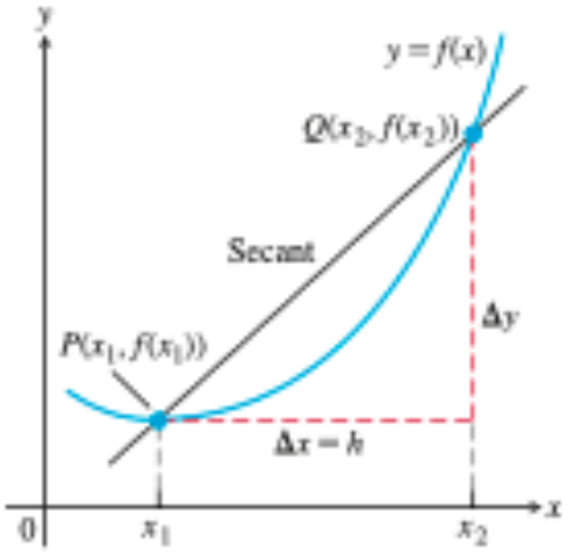


**FIGURE 2.1** A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of $f$ over the interval $[x_1, x_2]$.

**DEFINITION** The **average rate of change** of $y = f(x)$ with respect to $x$ over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of $f$ over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of $f$ from $x_1$ to $x_2$ is identical with the slope of secant line $PQ$. As the point $Q$ approaches the point $P$ along the curve, the length $h$ of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

**EXAMPLE 3** Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

**Solution** We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line $PQ$ and investigate what happens to the slope as $Q$ approaches $P$ along the curve:

$$\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then $Q$ lies above and to the right of $P$, as in Figure 2.4. If $h < 0$, then $Q$ lies to the left of $P$ (not shown). In either case, as $Q$ approaches $P$ along the curve, $h$ approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at $P$.

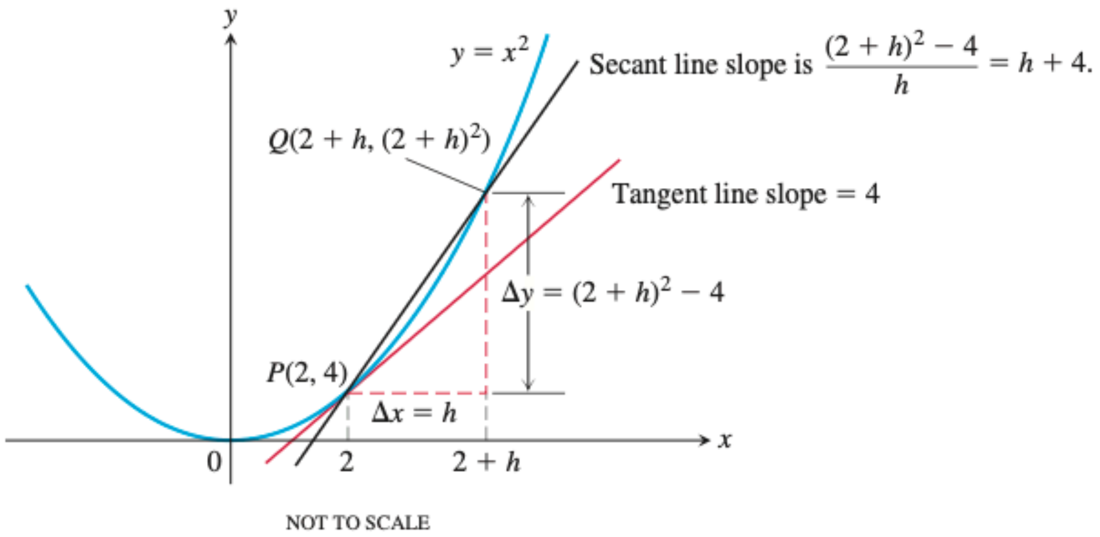


**FIGURE 2.4** Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

**EXAMPLE 3** Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

**Solution** We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line $PQ$ and investigate what happens to the slope as $Q$ approaches $P$ along the curve:

$$\begin{aligned}\text{Secant line slope} = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} &= \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, then $Q$ lies above and to the right of $P$, as in Figure 2.4. If $h < 0$, then $Q$ lies to the left of $P$ (not shown). In either case, as $Q$ approaches $P$ along the curve, $h$ approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at $P$.

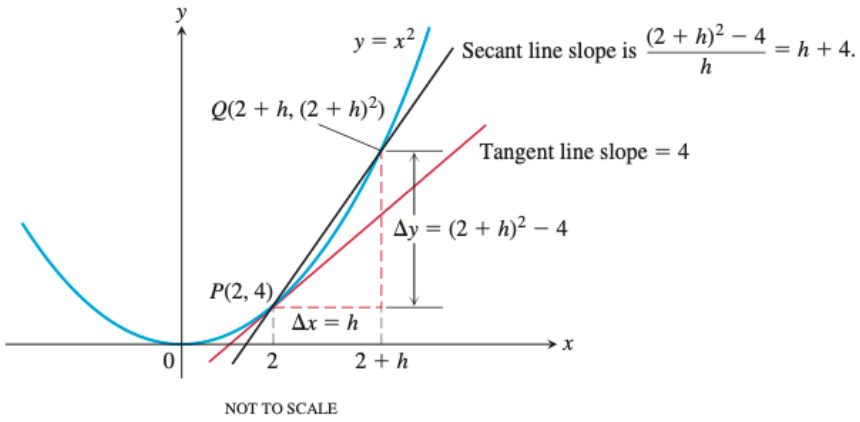


**FIGURE 2.4** Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at $P$ is the line through $P$ with slope 4:

$$\begin{aligned} y &= 4 + 4(x - 2) \qquad \text{Point-slope equation} \\ y &= 4x - 4. \end{aligned}$$

### Slope of a Curve at a Point

In Exercises 7–18, use the method in Example 3 to find **(a)** the slope of the curve at the given point $P$, and **(b)** an equation of the tangent line at $P$.

**7.** $y = x^2 - 5, \quad P(2, -1)$

**8.** $y = 7 - x^2, \quad P(2, 3)$

**9.** $y = x^2 - 2x - 3, \quad P(2, -3)$

**10.** $y = x^2 - 4x, \quad P(1, -3)$

**11.** $y = x^3, \quad P(2, 8)$

**12.** $y = 2 - x^3, \quad P(1, 1)$

**13.** $y = x^3 - 12x, \quad P(1, -11)$

**14.** $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

**15.** $y = \dfrac{1}{x}, \quad P(-2, -1/2)$

## An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function $f$ at an interior point of the domain of $f$. Suppose that $f(x)$ is defined on an open interval about $c$, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number $L$ (as close to $L$ as we like) for all $x$ sufficiently close to $c$, other than $c$ itself, then we say that $f$ approaches the **limit** $L$ as $x$ approaches $c$, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as $x$ approaches $c$ is $L$." In Example 1 we would say that $f(x)$ approaches the *limit* 2 as $x$ approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number $L$ whenever $x$ is close to $c$. The value of the function at $c$ itself is not considered.

**TABLE 2.2** As $x$ gets closer to 1, $f(x)$ gets closer to 2.

| $x$ | $f(x) = \frac{x^2-1}{x-1}$ |
|---|---|
| 0.9 | 1.9 |
| 1.1 | 2.1 |
| 0.99 | 1.99 |
| 1.01 | 2.01 |
| 0.999 | 1.999 |
| 1.001 | 2.001 |
| 0.999999 | 1.999999 |
| 1.000001 | 2.000001 |

## The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

### THEOREM 1—Limit Laws

If $L$, $M$, $c$, and $k$ are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, $n$ a positive integer

(If $n$ is even, we assume that $f(x) \geq 0$ for $x$ in an interval containing $c$.)

**EXAMPLE 5** Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

**(a)** $\displaystyle\lim_{x\to c}(x^3 + 4x^2 - 3)$

**(b)** $\displaystyle\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$

**(c)** $\displaystyle\lim_{x\to -2}\sqrt{4x^2 - 3}$

**EXAMPLE 5** Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

**(a)** $\lim\limits_{x\to c}(x^3 + 4x^2 - 3)$

**(b)** $\lim\limits_{x\to c}\dfrac{x^4 + x^2 - 1}{x^2 + 5}$

**(c)** $\lim\limits_{x\to -2}\sqrt{4x^2 - 3}$

**Solution**

**(a)**

$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules}\end{aligned}$$

**(b)**

$$\begin{aligned}\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim\limits_{x\to c}(x^4 + x^2 - 1)}{\lim\limits_{x\to c}(x^2 + 5)} && \text{Quotient Rule}\\ &= \frac{\lim\limits_{x\to c} x^4 + \lim\limits_{x\to c} x^2 - \lim\limits_{x\to c} 1}{\lim\limits_{x\to c} x^2 + \lim\limits_{x\to c} 5} && \text{Sum and Difference Rules}\\ &= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}\end{aligned}$$

**(c)**

$$\begin{aligned}\lim_{x\to -2}\sqrt{4x^2 - 3} &= \sqrt{\lim_{x\to -2}(4x^2 - 3)} && \text{Root Rule with } n = 2\\ &= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3} && \text{Difference Rule}\\ &= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules and limit of a constant function}\\ &= \sqrt{16 - 3}\\ &= \sqrt{13}\end{aligned}$$

■

## Calculating Limits

Find the limits in Exercises 11–22.

**11.** $\lim_{x \to -3} (x^2 - 13)$

**12.** $\lim_{x \to 2} (-x^2 + 5x - 2)$

**13.** $\lim_{t \to 6} 8(t - 5)(t - 7)$

**14.** $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$

**15.** $\lim_{x \to 2} \frac{2x + 5}{11 - x^3}$

**16.** $\lim_{s \to 2/3} (8 - 3s)(2s - 1)$

**17.** $\lim_{x \to -1/2} 4x(3x + 4)^2$

**18.** $\lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6}$

**19.** $\lim_{y \to -3} (5 - y)^{4/3}$

**20.** $\lim_{z \to 4} \sqrt{z^2 - 10}$

- Questions?

Study hard, best of luck!